Manifold Sampling for Piecewise Linear Nonconvex Optimization

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Problem statement

We are interested in solving the problem:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \triangleq \psi(x) + h(F(x))$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^p \to \mathbb{R}$,

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Piecewise linear h does not imply $h \circ F$ is piecewise linear.

Notes

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- We will build component models m^{F_i} of each F_i around points x. We can then use $\nabla M(x) \in \mathbb{R}^{n\times p}$ where

$$\nabla M(x) \triangleq \left[\nabla m^{F_1}(x), \dots, \nabla m^{F_p}(x)\right].$$

Piecewise linear functions

Definition

A function $h\colon \mathbb{R}^p \to \mathbb{R}$ is *piecewise linear* if h is continuous and there exists a finite collection $\mathfrak{H} \triangleq \{h_i : i = 1, \ldots, \hat{m}\}$ of affine functions that map $\mathbb{R}^p$ into $\mathbb{R}$, for which

$$h(z) \in \left\{\tilde{h}(z) : \tilde{h} \in \mathfrak{H}\right\}, \qquad \forall z \in \mathbb{R}^p.$$

- h is a *continuous selection* of $\mathfrak{H}$.
- Elements of $\mathfrak{H}$ are *selection functions of* h.
- $h_i : z \in \mathbb{R}^p \mapsto \langle a_i, z\rangle + b_i$ for each i.

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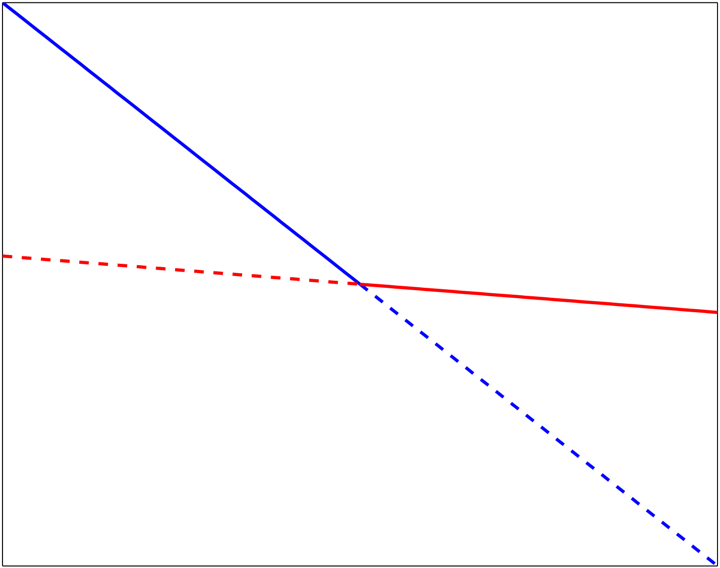
Definition

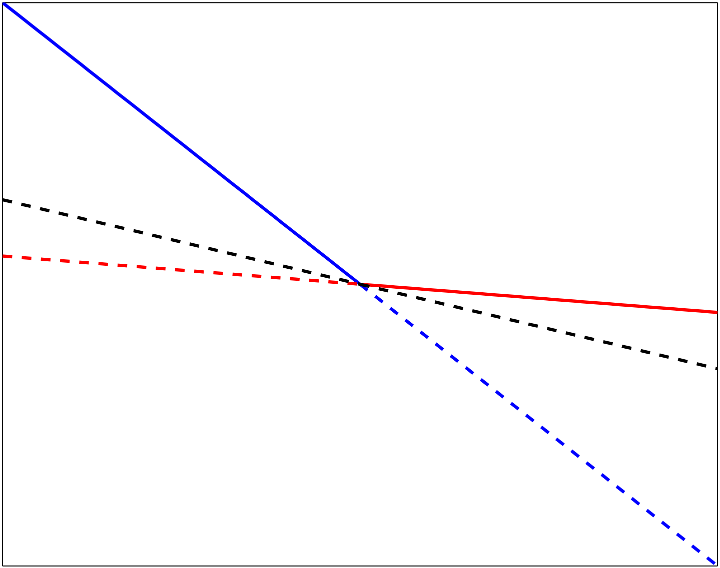
$$\mathcal{S}_i \triangleq \{y : h(y) = h_i(y)\}, \quad \tilde{\mathcal{S}}_i \triangleq \mathbf{cl}\left(\mathbf{int}\left(\mathcal{S}_i\right)\right), \quad I_h(z) \triangleq \left\{i : z \in \tilde{\mathcal{S}}_i\right\},$$

h_i for $i \in I_h(z)$ is an *essentially active selection function* for h at z.

Essentially active

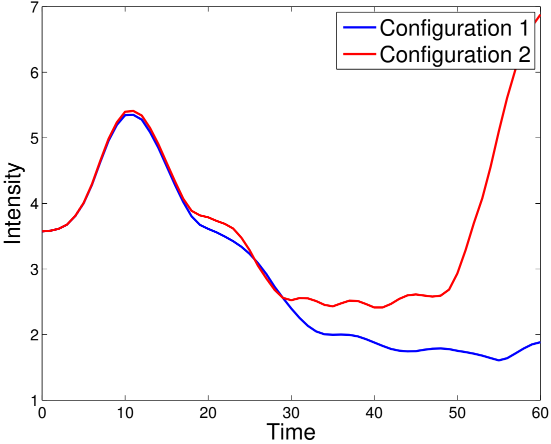


Essentially active



Laser pulse propagating in a plasma channel

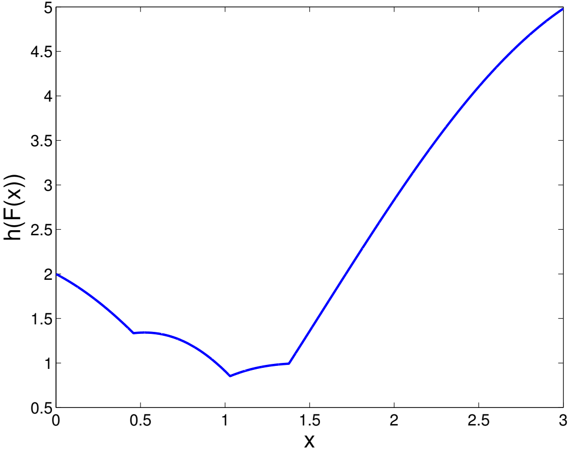
Determine plasma channel properties that minimize the maximum difference in the laser intensity.



$$f(x) = \max_{\Omega_1} \{F_i(x)\} - \min_{\Omega_2} \{F_i(x)\}$$

Formulation

$$h(F(x)) = \max\{\sin(2x)+1, \cos(2x), x\} - \min\{\sin(2x)+1, \cos(2x), x\}$$



A generalized derivative

Definition

The *generalized Clarke subdifferential* of f at x is defined as

$$\partial_{\mathrm{C}} f(x) \triangleq \mathbf{co}\left(\left\{\xi : \xi = \lim_{y^j \to x} \nabla f(y^j) : \; y^j \in \mathcal{D}\right\}\right),$$

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Definition

A point x is called a *Clarke stationary* point of f if $0 \in \partial_{\mathrm{C}} f(x)$.

Algorithm components

- Generator set $\mathfrak{G}^k$

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Generator set

At some iterate x^k,

$$\mathfrak{G}^k \triangleq \bigcup_{i \in I_h(F(x^k))} \left\{ \nabla \psi(x^k) + \nabla M(x^k) a_i \right\}$$

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Or, given a set of points $Y = \left\{x^k, y^2, \ldots, y^p\right\} \subset \mathcal{B}(x^k, \Delta_k)$,

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Assumption

The set $\mathfrak{G}^k$ satisfies

$$\{\nabla\psi(x^k) + \nabla M(x^k)\, a_i : i \in I_h(F(x^k))\} \subseteq \mathfrak{G}^k$$
$$\mathfrak{G}^k \subseteq \{\nabla\psi(x^k) + \nabla M(x^k)\, a_i : y \in \mathcal{B}\left(x^k; \Delta_k\right), i \in I_h(F(y))\}.$$

Smooth master model

Our model gradients around iterate x^k satisfy

$$g^k \triangleq \mathbf{proj}\left(0, \mathbf{co}\left(\mathfrak{G}^k\right)\right) \in \mathbf{co}\left(\mathfrak{G}^k\right),$$

Let λ^* be the corresponding coefficients so that $g^k = G^k \lambda^*$.

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Define

$$A^k \triangleq \begin{bmatrix} | & & | \\ a_{j_1} & \cdots & a_{j_t} \\ | & & | \end{bmatrix},$$

and set $w^k = A^k\lambda^*$. Define the smooth *master model* $m_k^f : \mathbb{R}^n \to \mathbb{R}$,

$$m_k^f(x) \triangleq \psi(x^k) + \sum_{i=1}^{p} w_i^k m^{F_i}(x) + \sum_{i=1}^{p} \lambda_i^* b_{j_i}.$$

Trust region subproblem

Approximately solve

$$\begin{aligned} &\underset{s}{\text{minimize}}\ m_k^f(x^k+s) \\ &\textbf{subject to: } s \in \mathcal{B}(0, \Delta_k) \end{aligned}$$

to obtain a solution s satisfying

$$\psi(x^k) - \psi(x^k+s) + \langle M(x^k) - M(x^k+s), w^k \rangle \geq \frac{\kappa_{\mathrm{d}}}{2} \|g^k\| \min\left\{\Delta_k, \frac{\|g^k\|}{\kappa_{\mathrm{mh}}}\right\}.$$

Measuring descent

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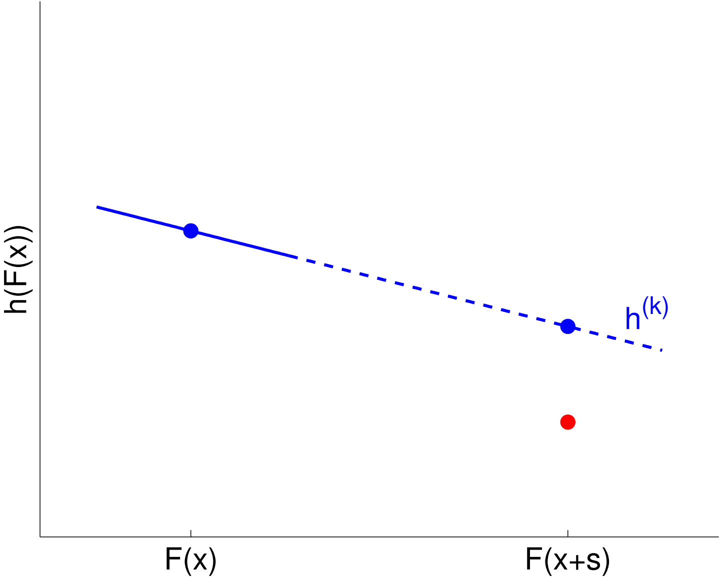
$$h^{(k)}(F(x^k)) \leq h(F(x^k)) \qquad \text{and} \qquad h^{(k)}(F(x^k+s^k)) \geq h(F(x^k+s^k)),$$

- $$\rho_k \triangleq \frac{\psi(x^k) - \psi(x^k+s^k) + h^{(k)}(F(x^k)) - h^{(k)}(F(x^k+s^k))}{\psi(x^k) - \psi(x^k+s^k) + \langle M(x^k) - M(x^k+s^k), a^{(k)} \rangle}$$

Examples of $h^{(k)}$



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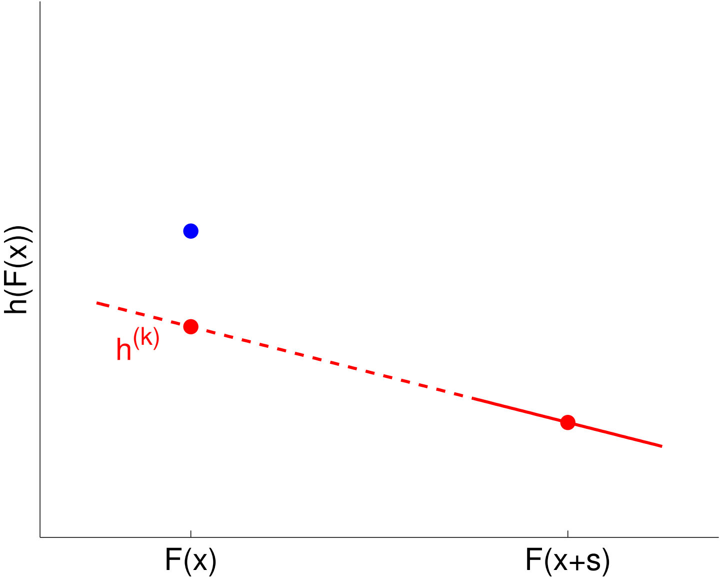

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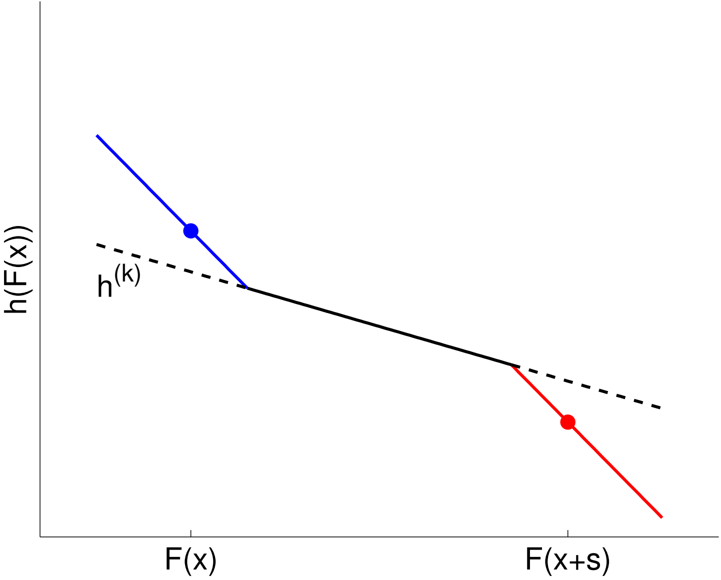

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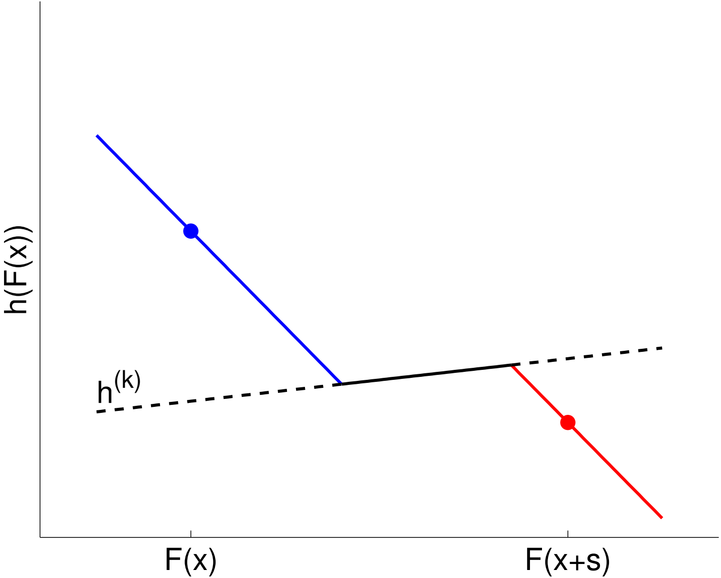

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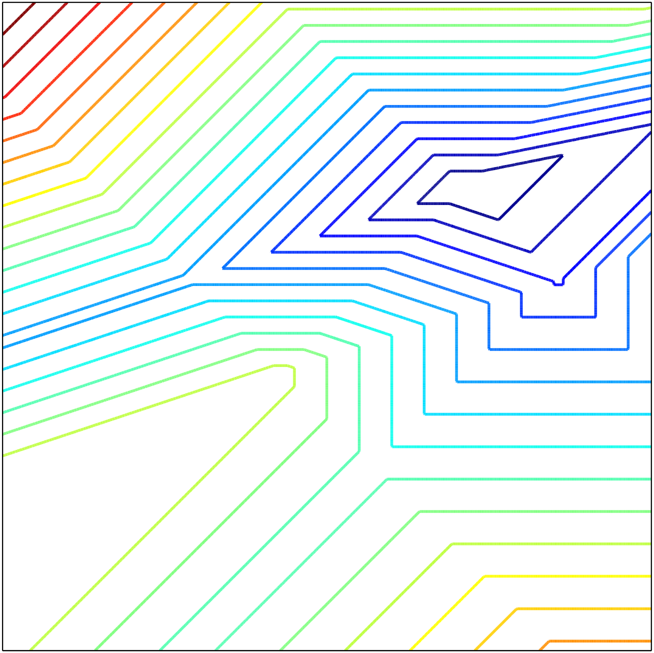
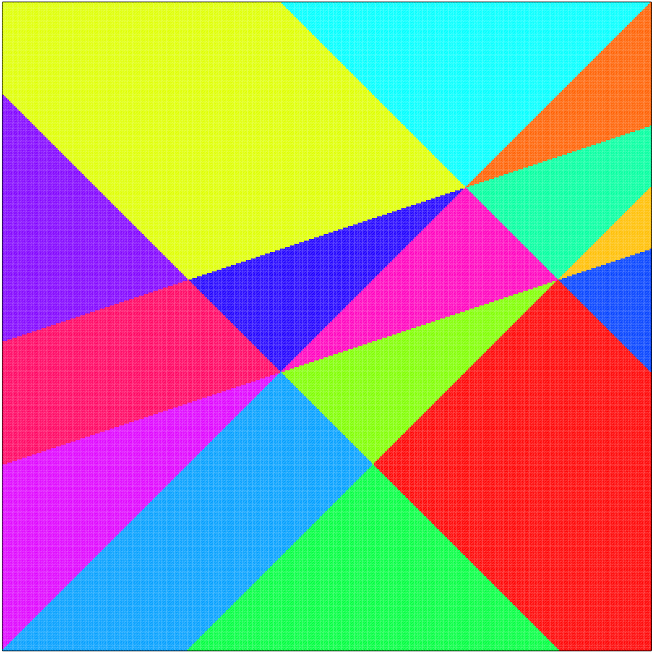
Algorithm MS4PL

Choose initial iterate x^0 and trust-region radius $\Delta_0 > 1$

for $k = 0, 1, 2, \ldots$ **do**

- Build p component models m^{F_i} that are fully linear on $\mathcal{B}(x^k, \Delta_k)$
- Form $\nabla M(x^k)$ using $\nabla m^{F_i}(x^k)$ and construct $\mathfrak{G}^k \subset \mathbb{R}^n$
- $\rho_k \leftarrow -\infty$
- **while** $\rho_k = -\infty$ **do**
  - Update component models m^{F_i}; build master model m^f
  - **if** $\Delta_k < \eta_2 \|\nabla m^f(x^k)\|$ (*acceptability* criterion) **then**
    - Approximately solve TRSP to obtain s^k
    - Evaluate $F(x^k + s^k)$ and find $h^{(k)}$
    - **if** $(\nabla \psi(x^k) + \nabla M(x^k)\, a^{(k)}) \in \mathfrak{G}^k$ **then**
      - Calculate ρ_k
    - **else**
      - $\mathfrak{G}^k \leftarrow \mathfrak{G}^k \cup \{\nabla \psi(x^k) + \nabla M(x^k)\, a^{(k)}\}$
  - **else**
    - break out of while-loop; iteration is *unacceptable*
- **if** $\rho_k > \eta_1 > 0$ (*successful* iteration) **then**
  - $x^{k+1} \leftarrow x^k + s^k$, $\Delta_{k+1} \leftarrow \min\{\gamma_{\mathrm{inc}}\Delta_k, \Delta_{\max}\}$
- **else**
  - $x^{k+1} \leftarrow x^k$, $\Delta_{k+1} \leftarrow \gamma_{\mathrm{dec}}\Delta_k$





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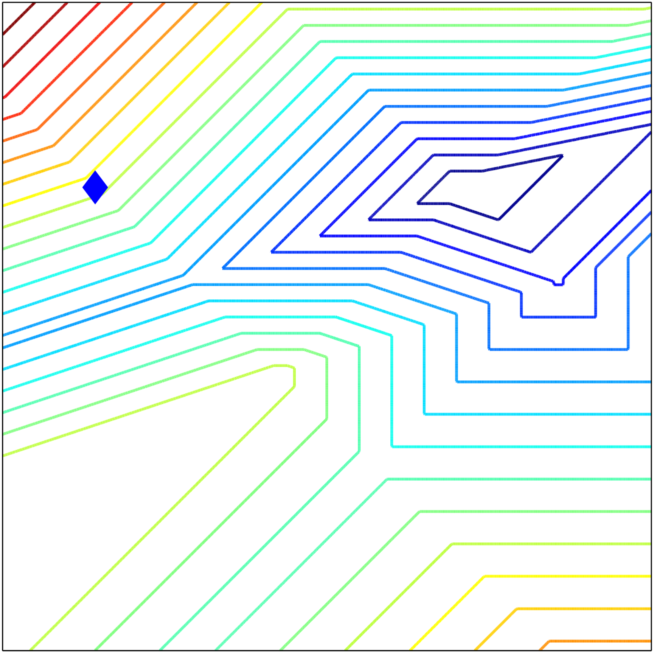
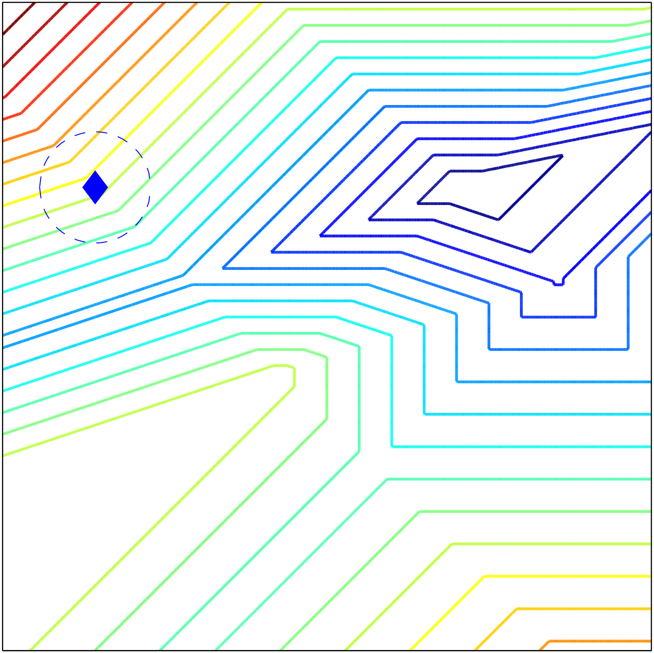
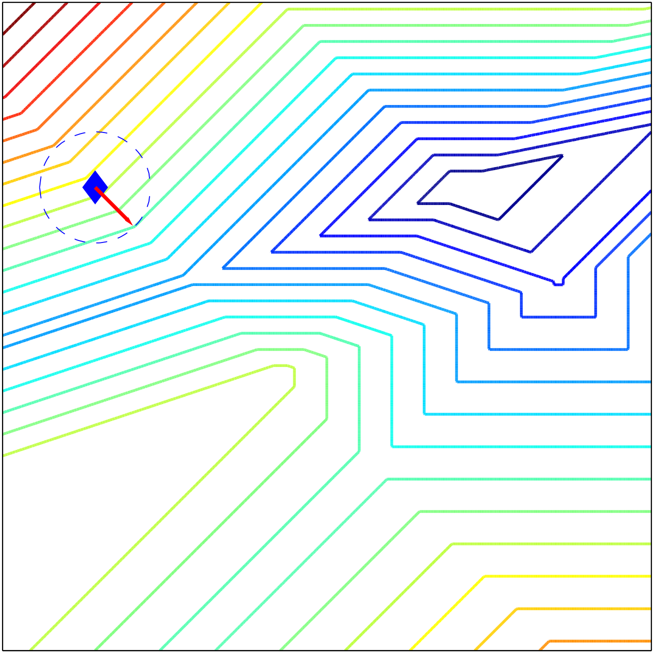
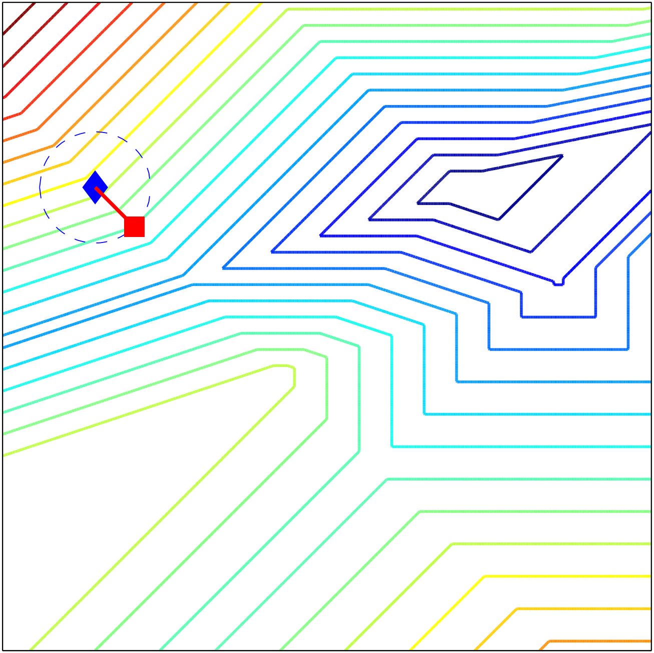
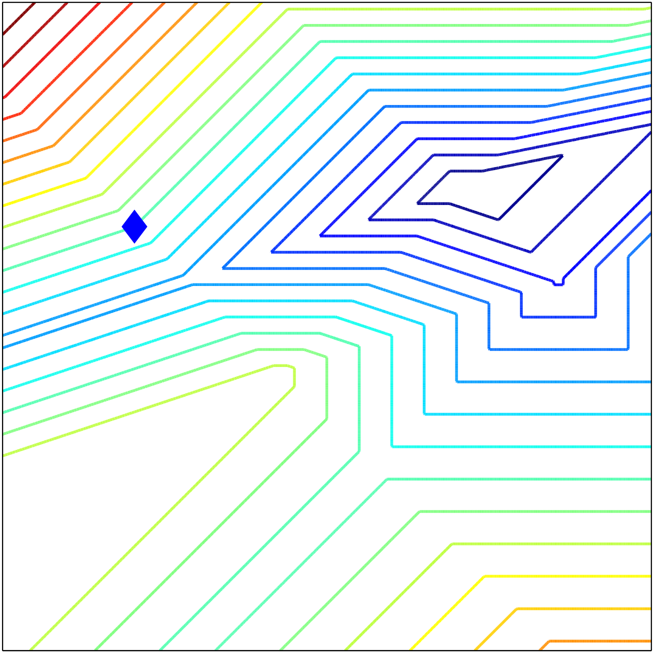
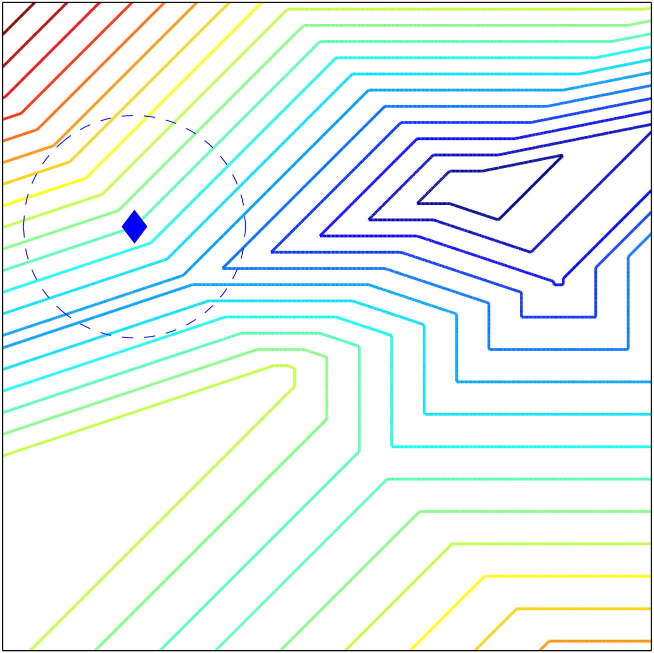


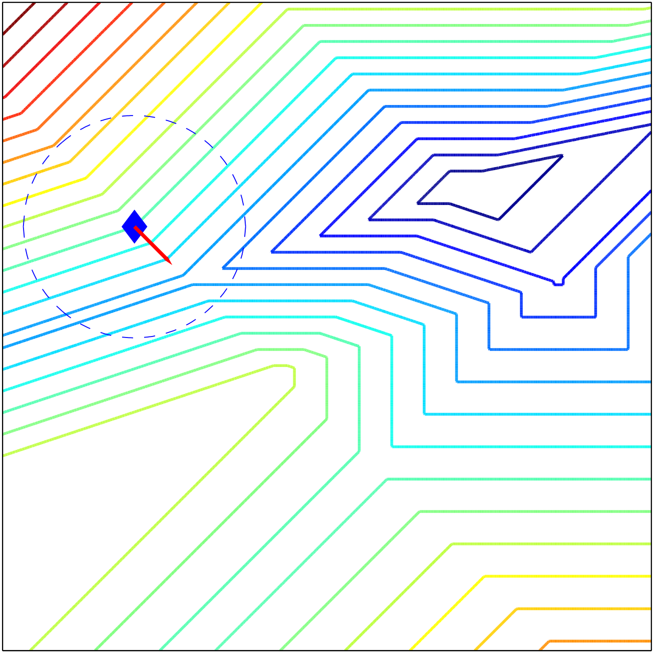


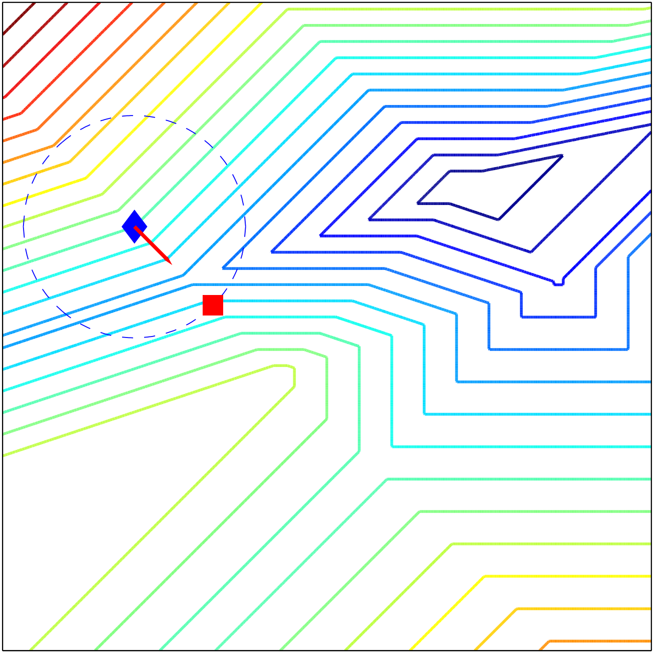


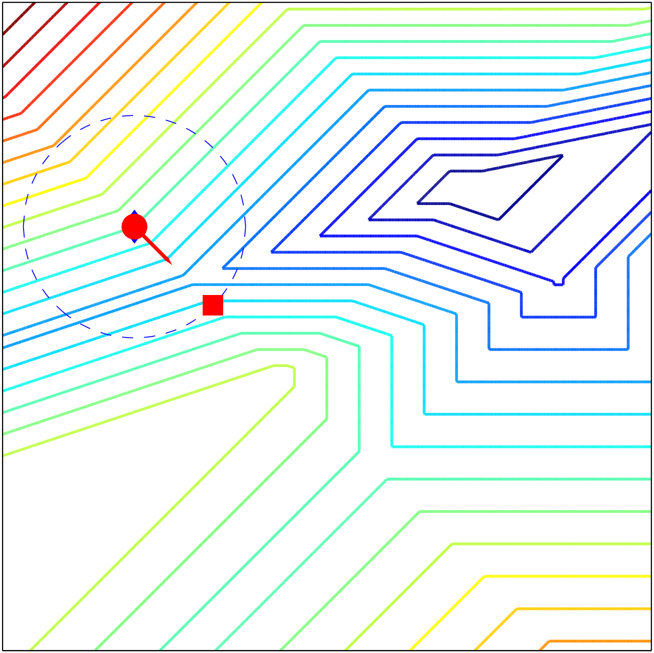


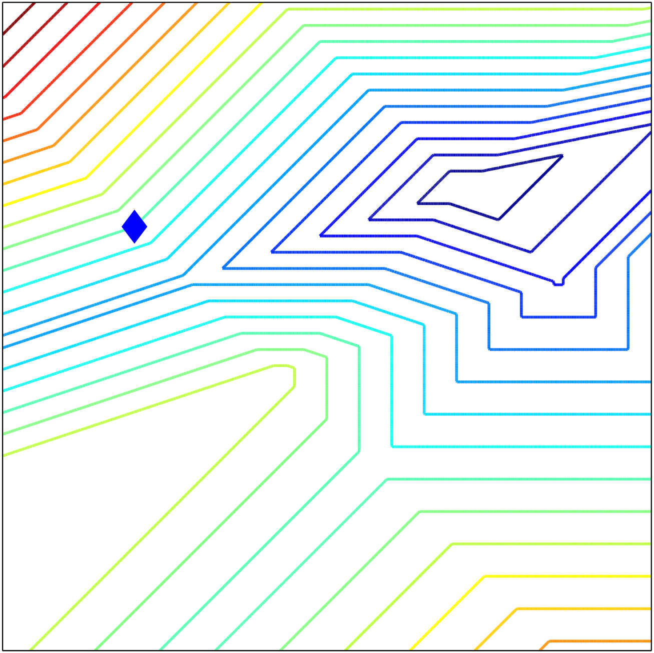


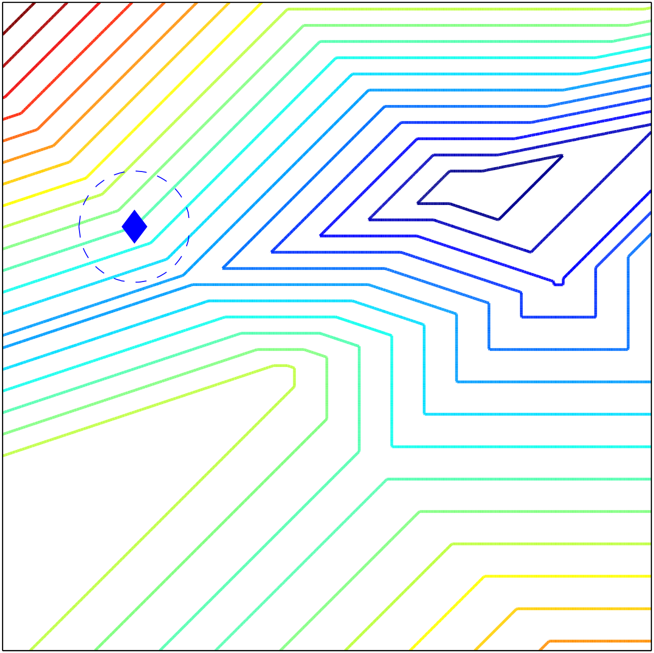


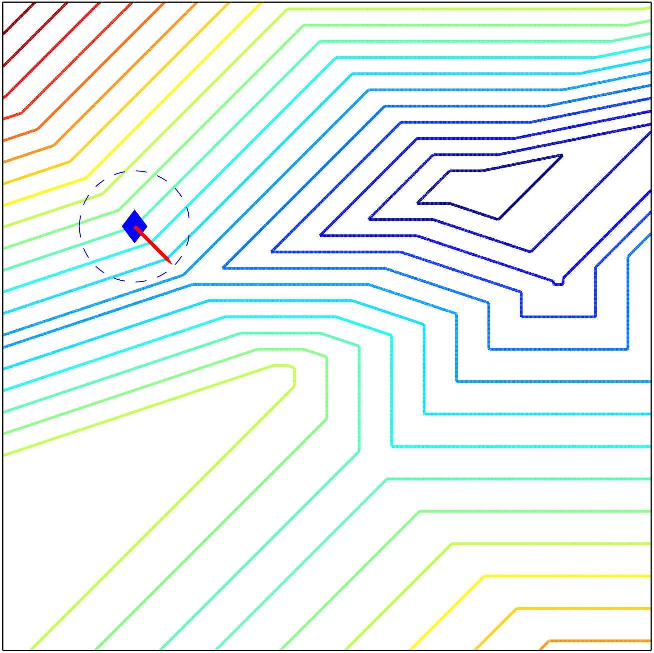


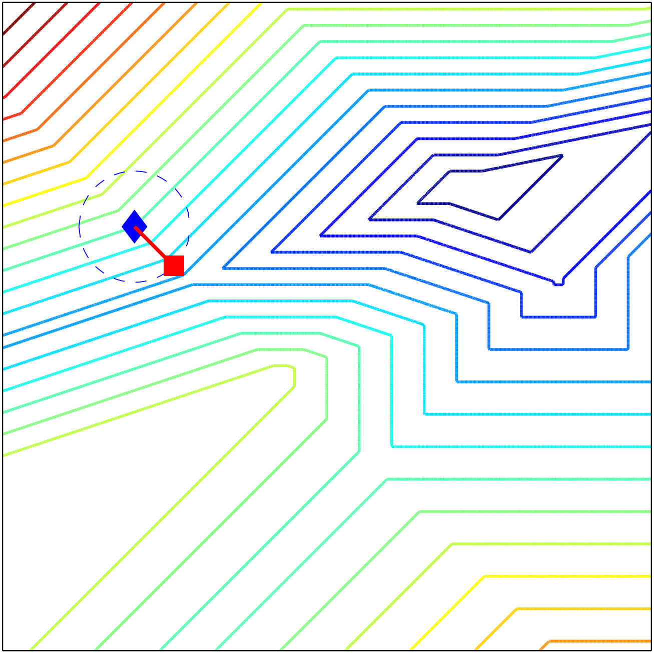


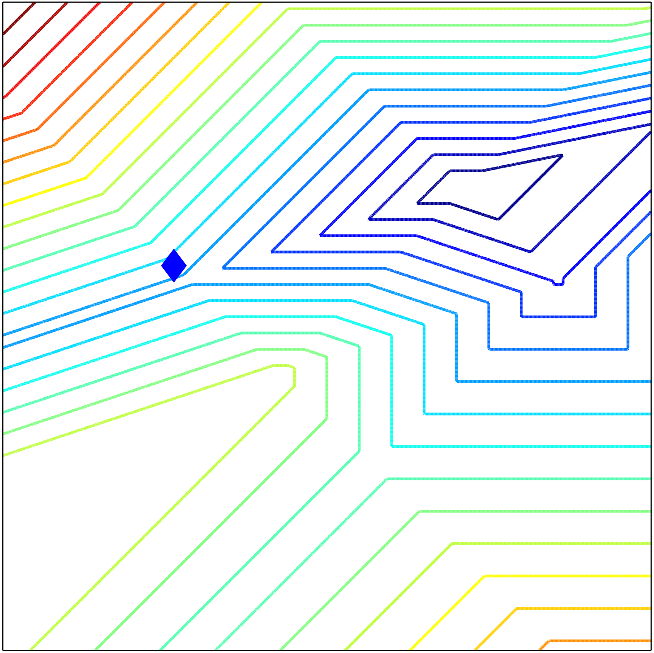


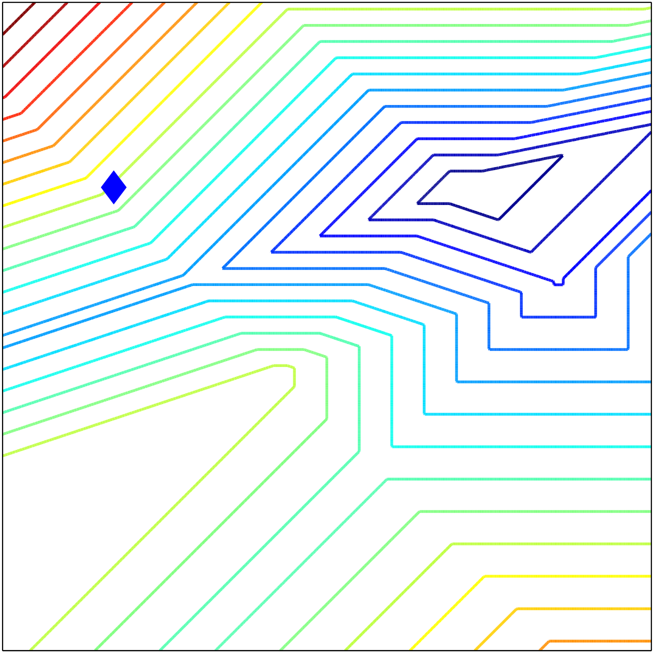


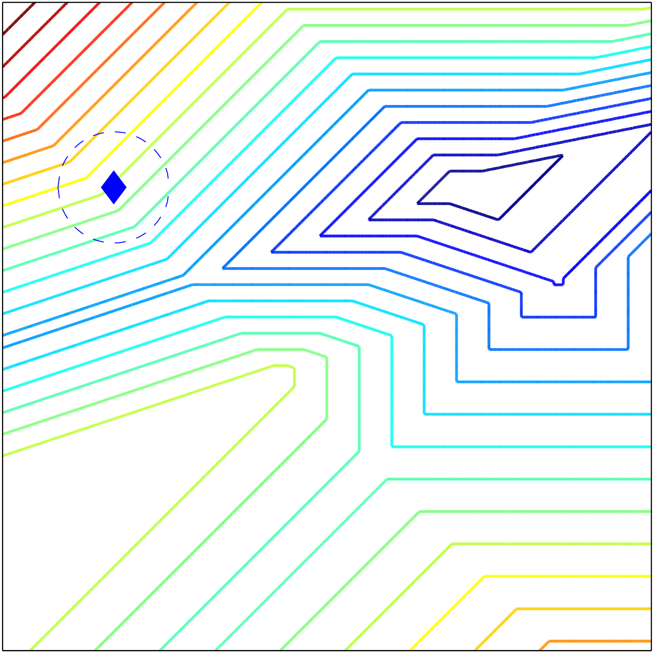


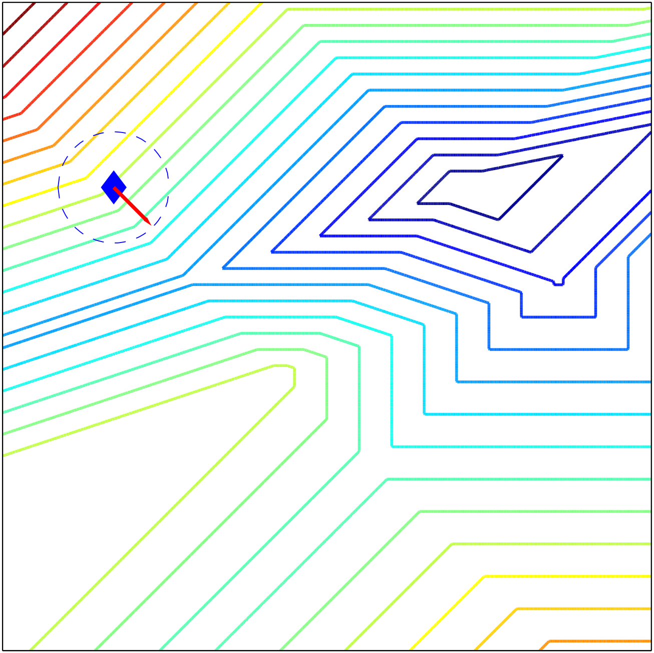


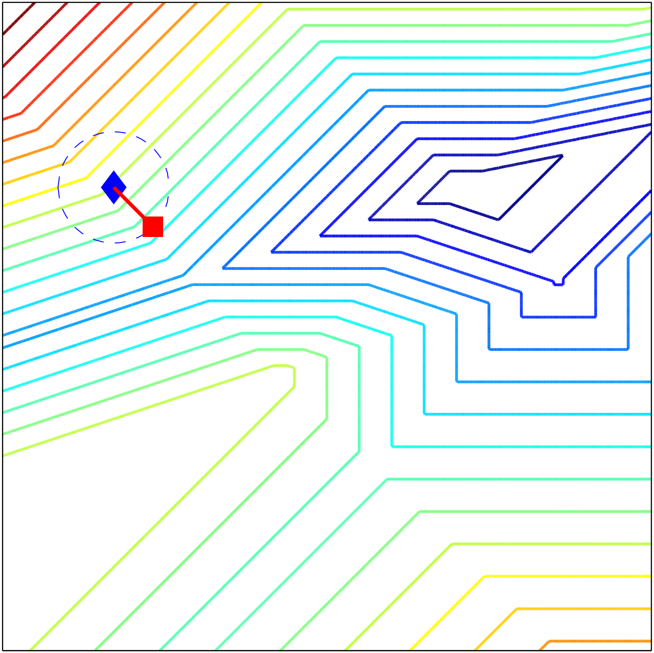


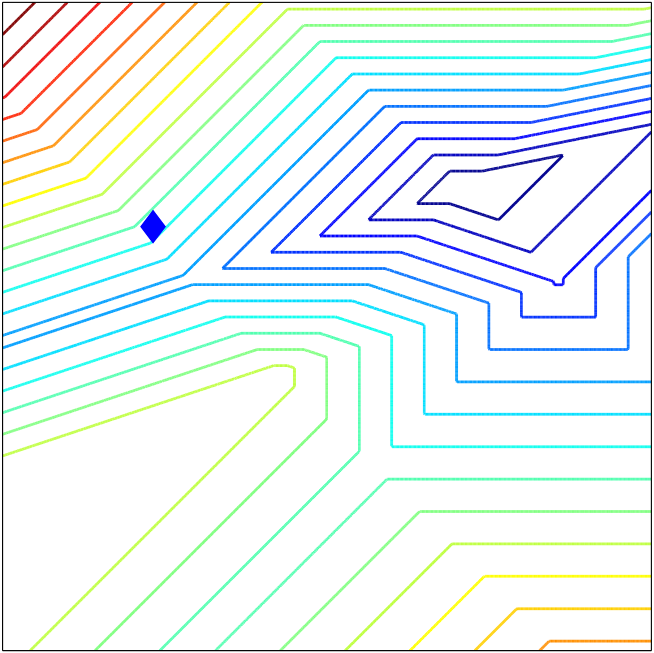


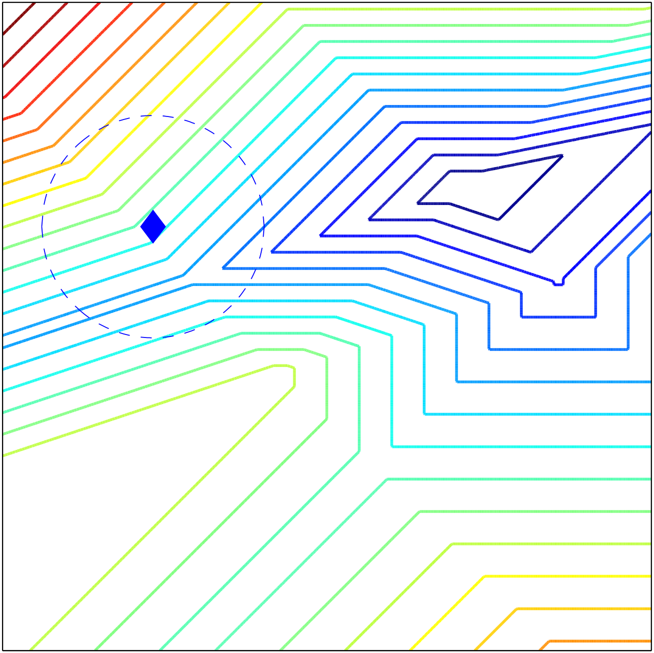


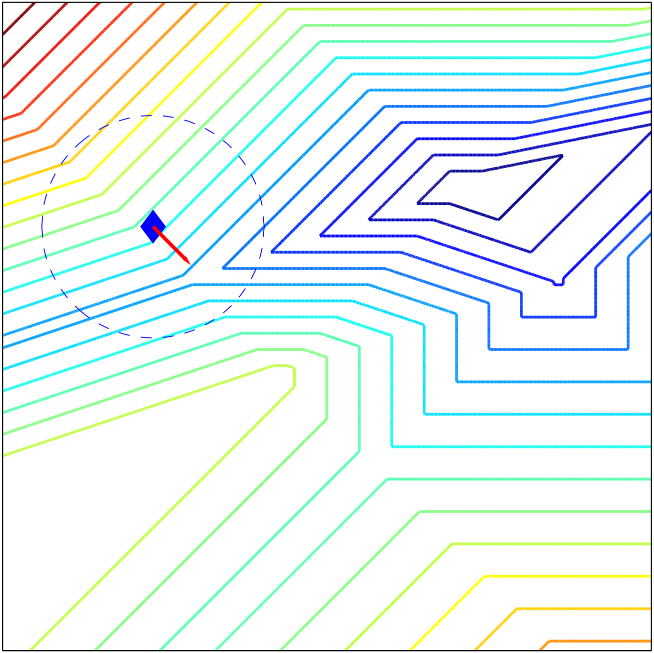


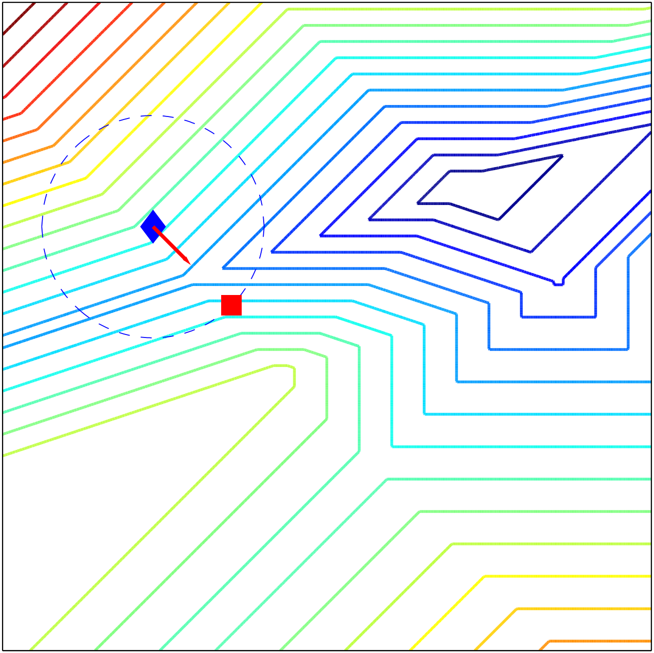


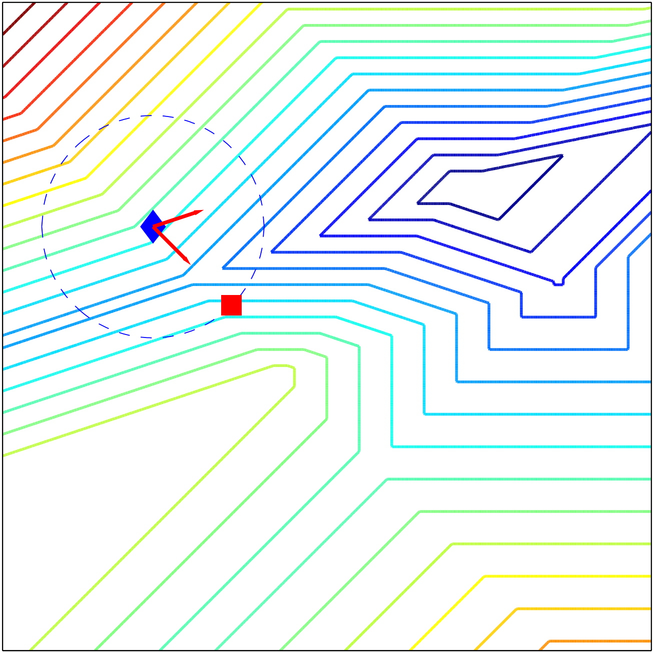


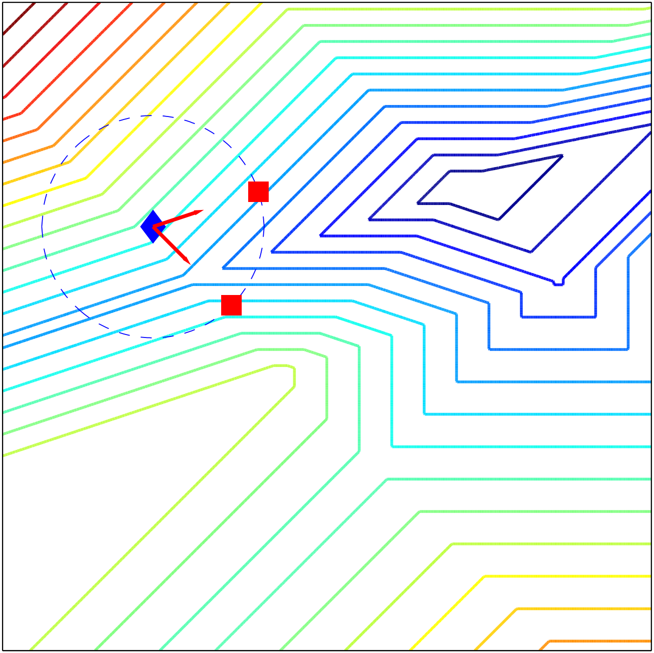


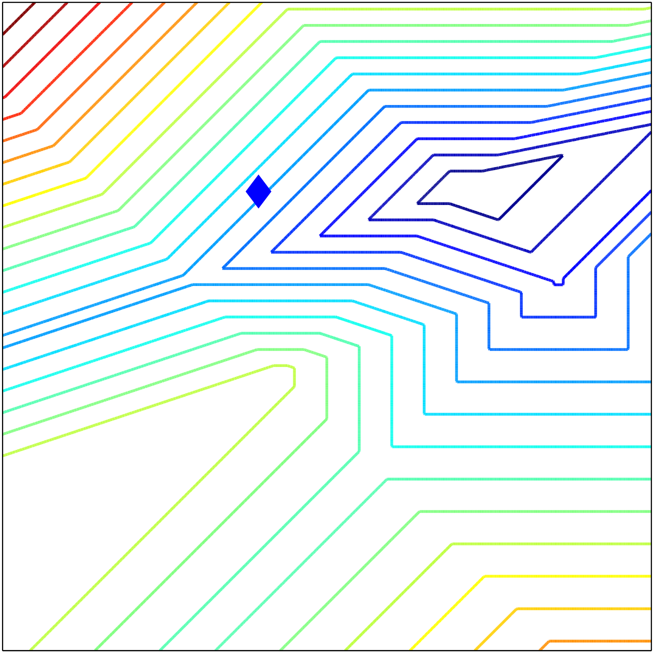


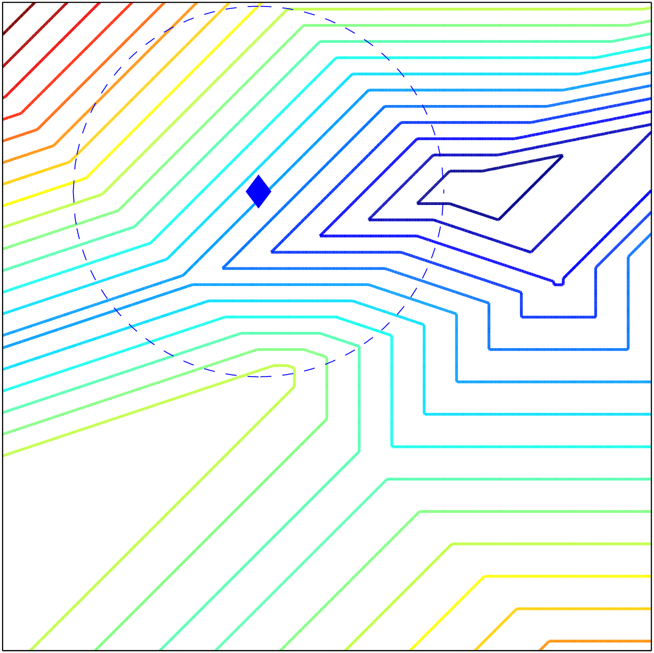


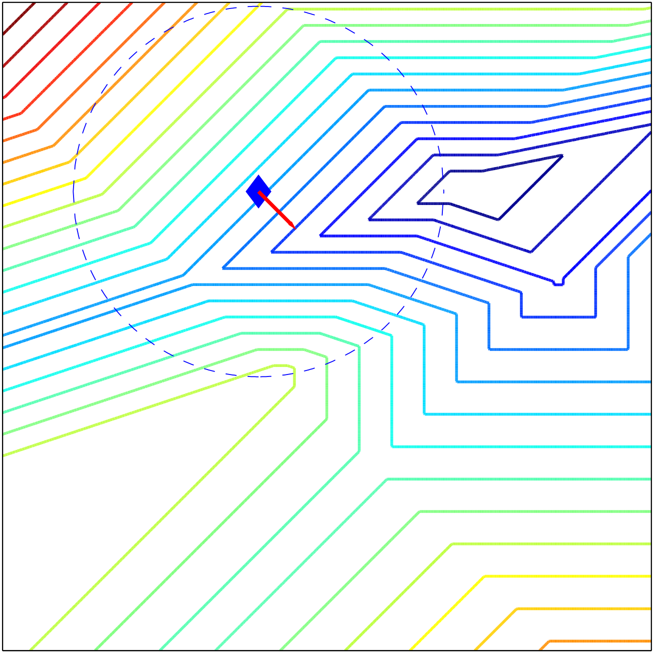


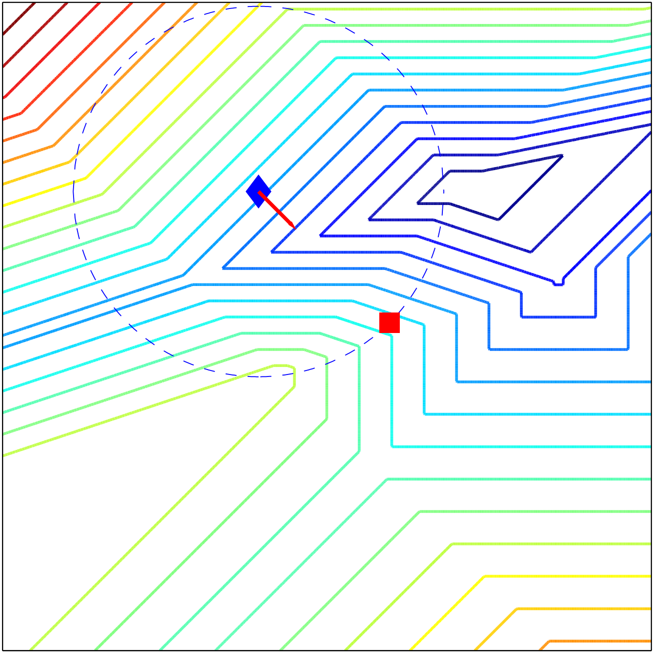


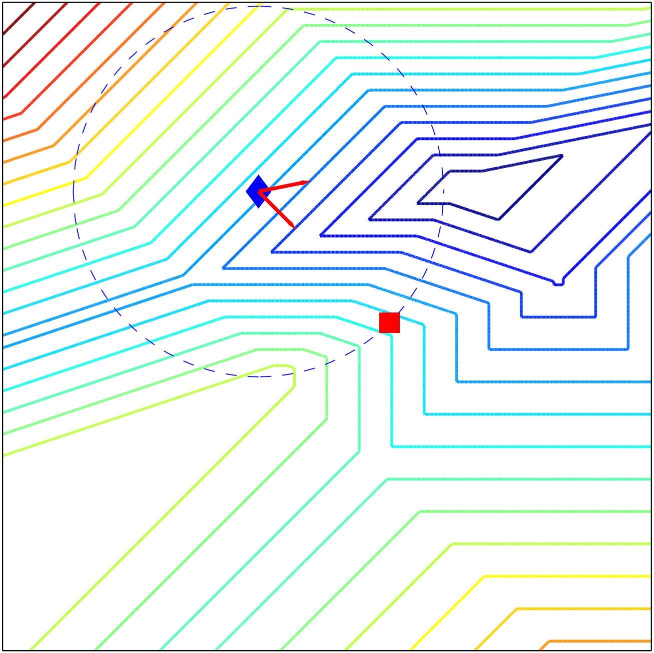


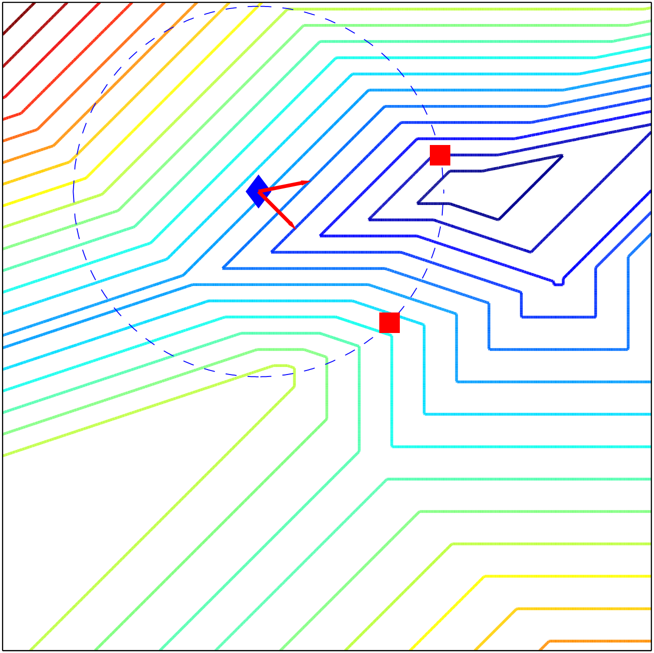


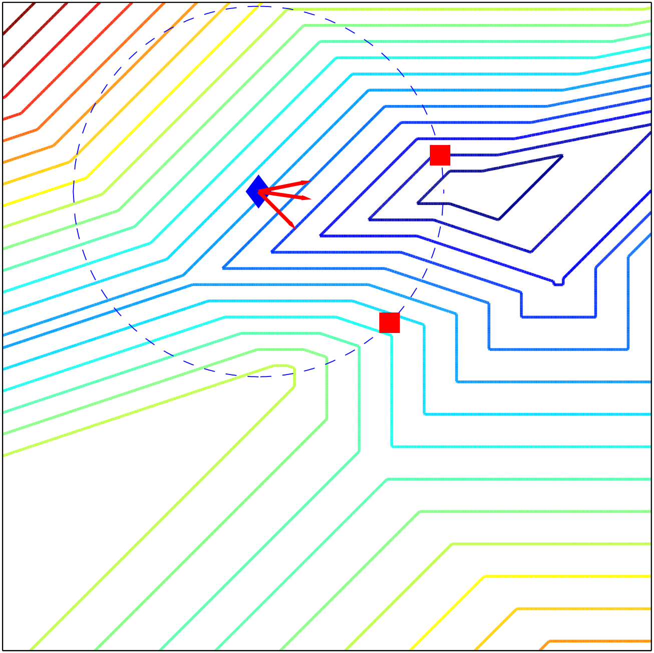


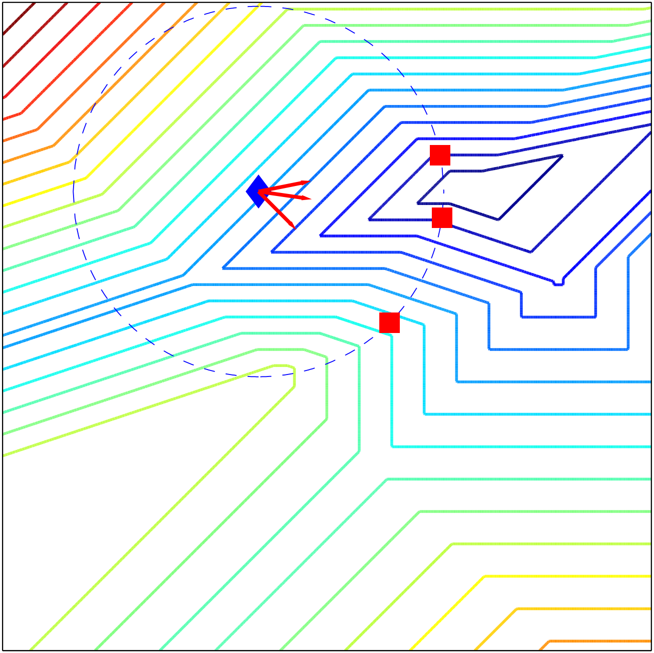


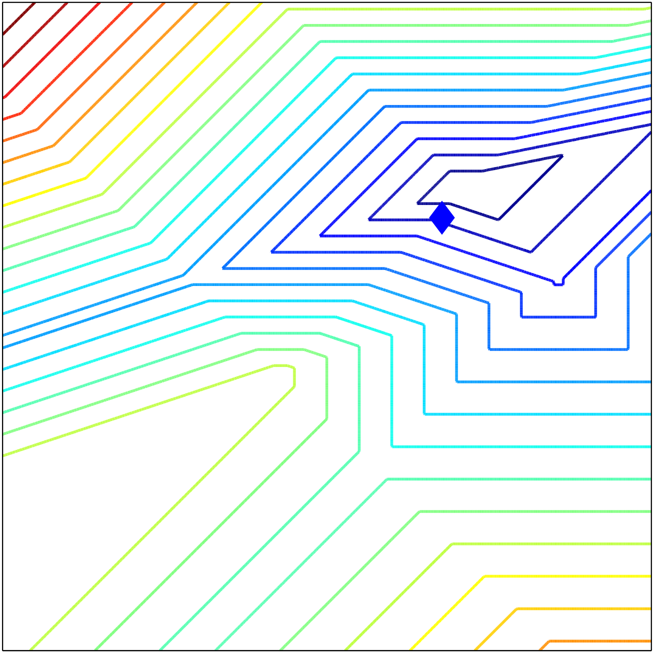


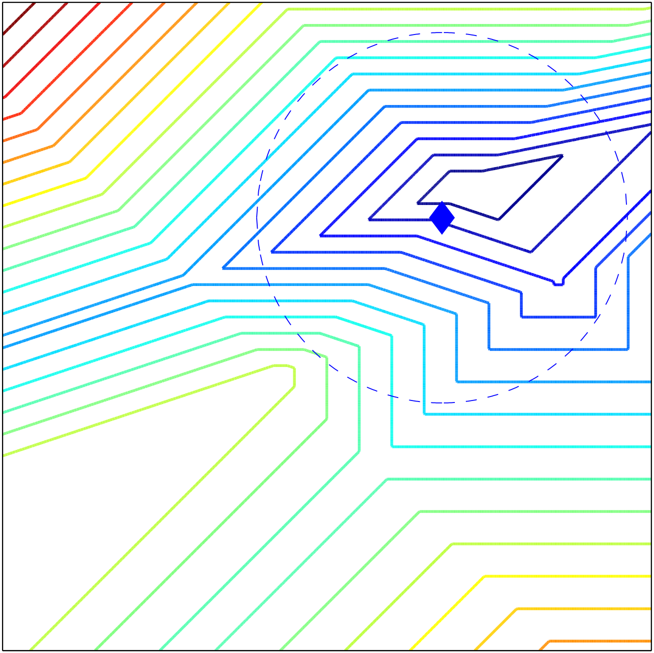


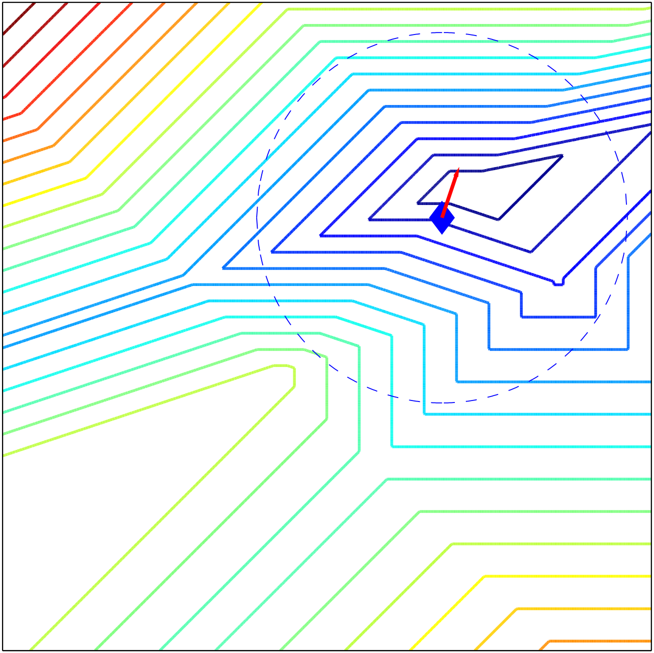


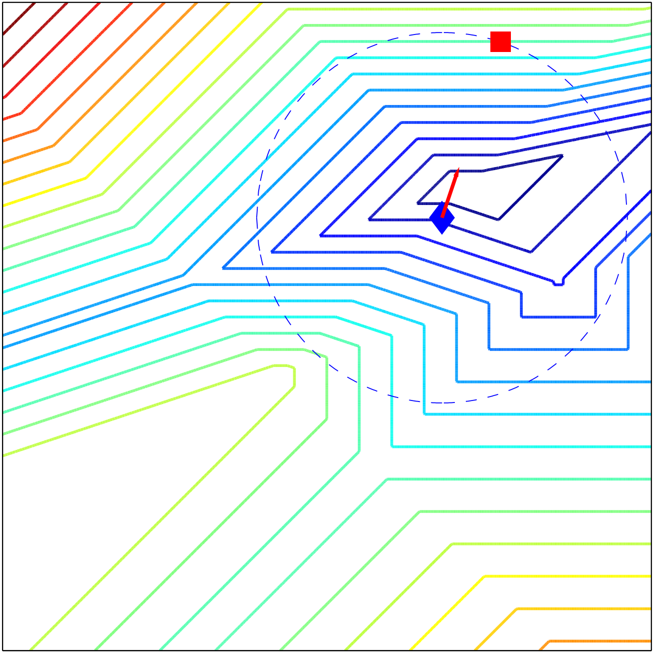


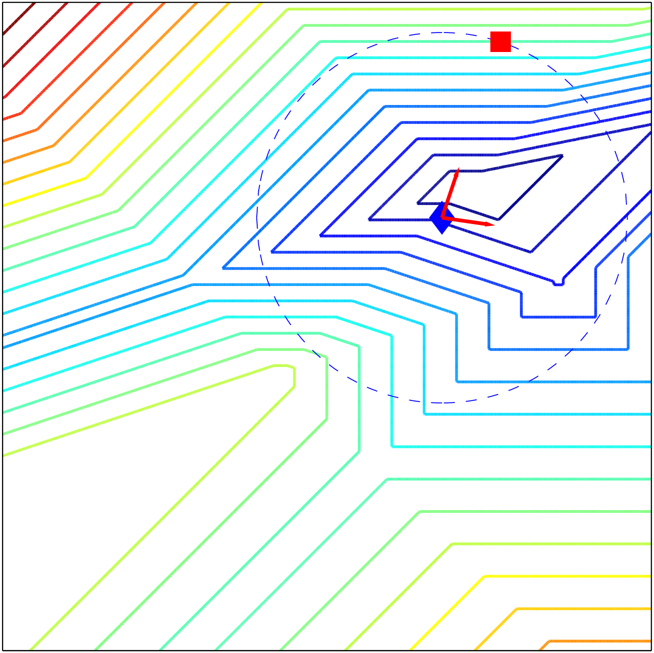


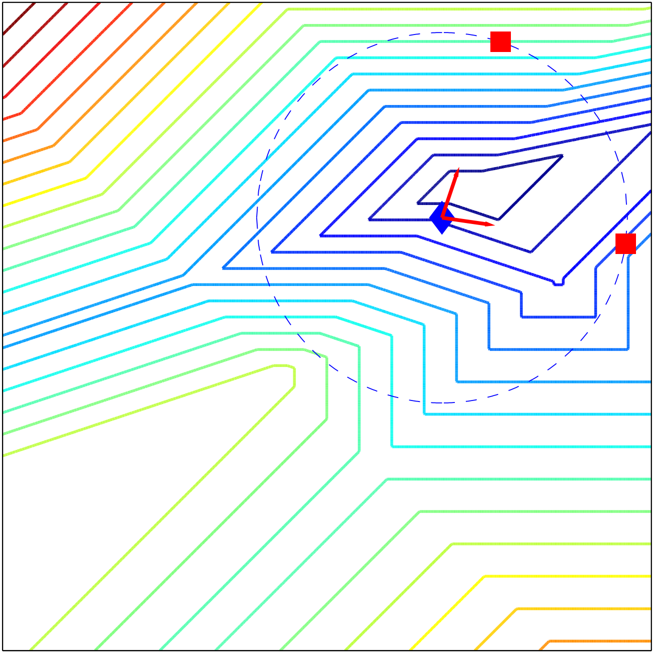


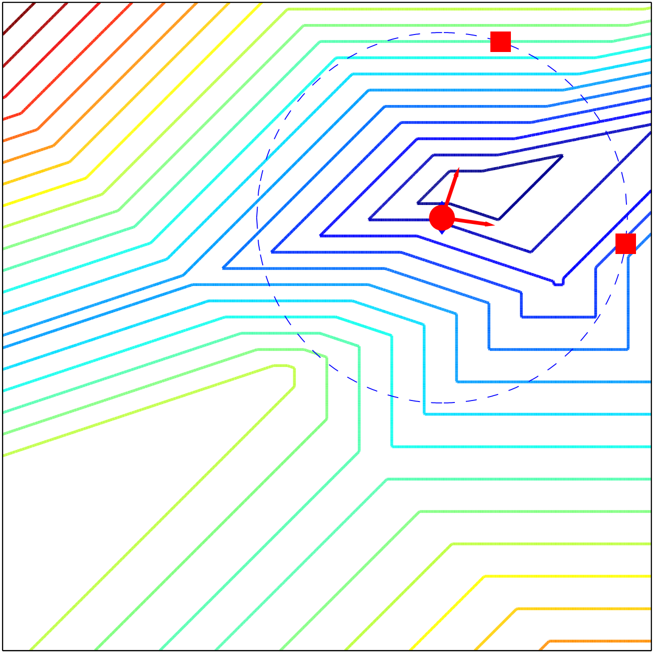


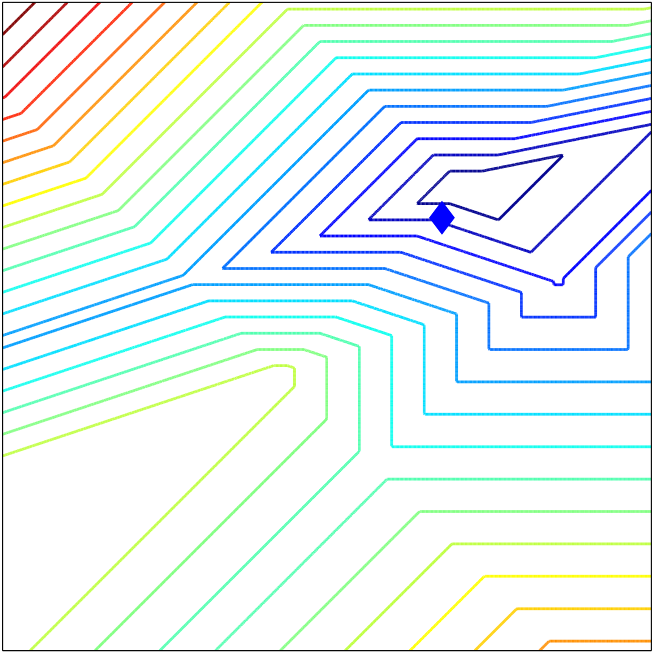


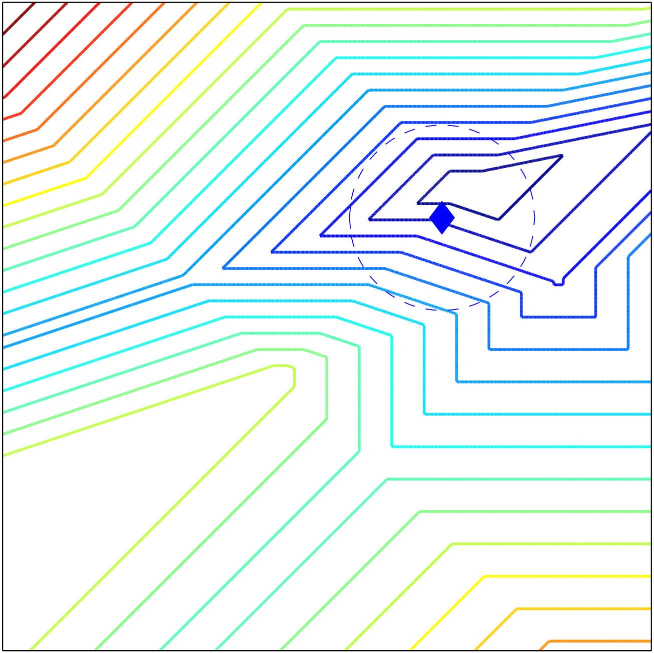


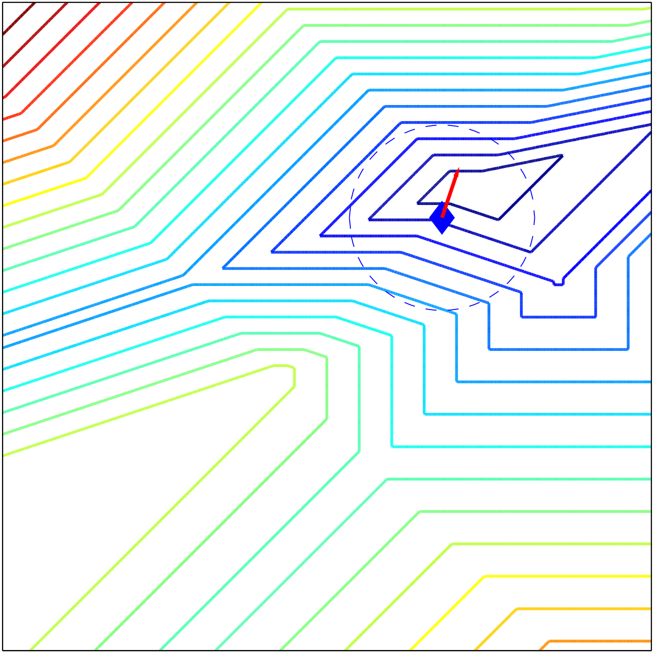


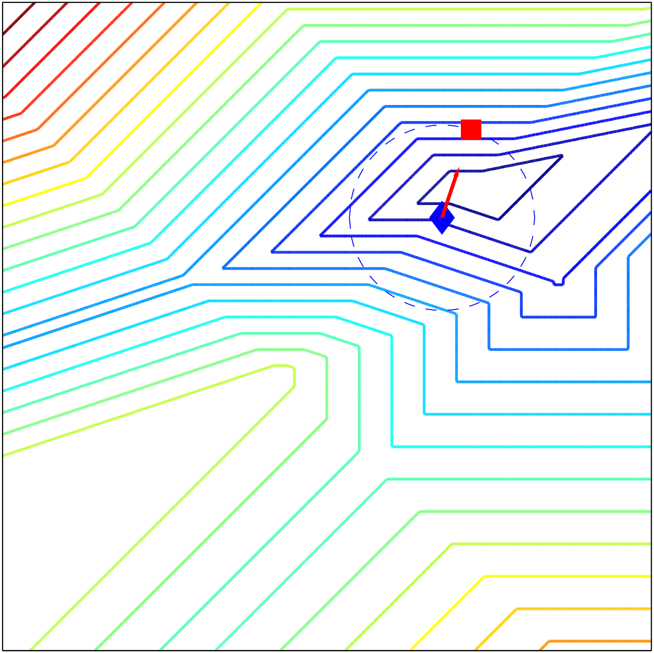


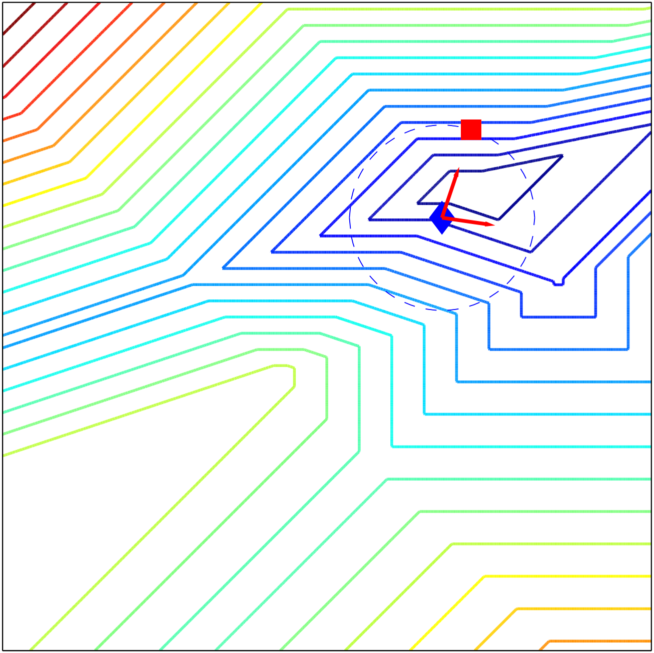


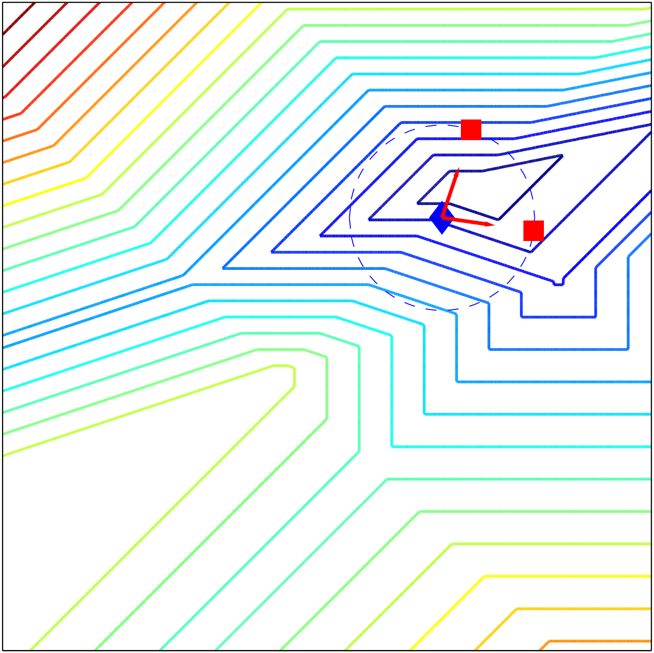


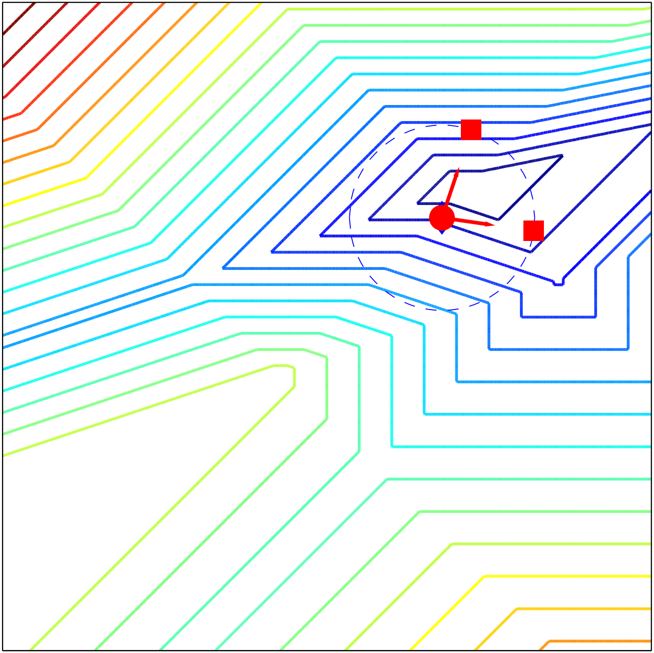


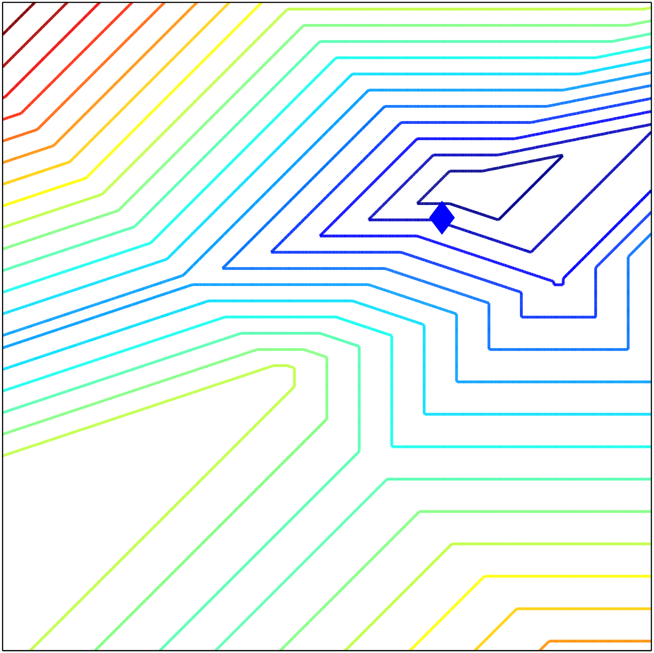


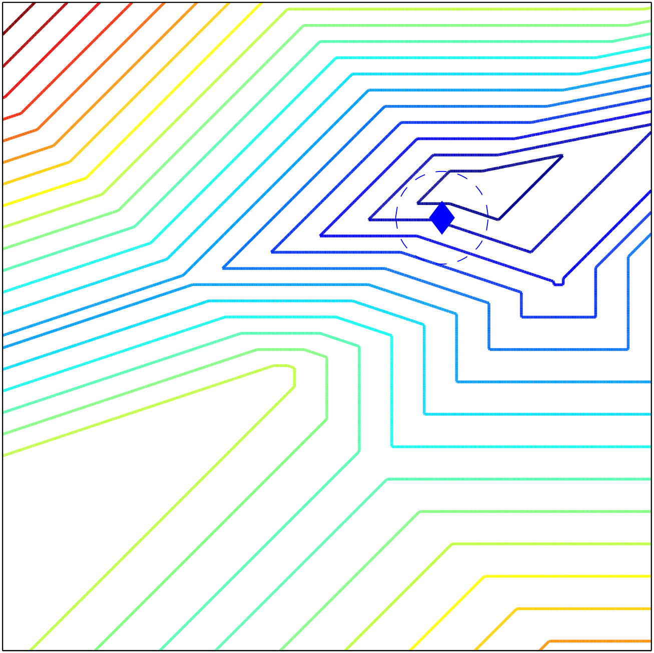


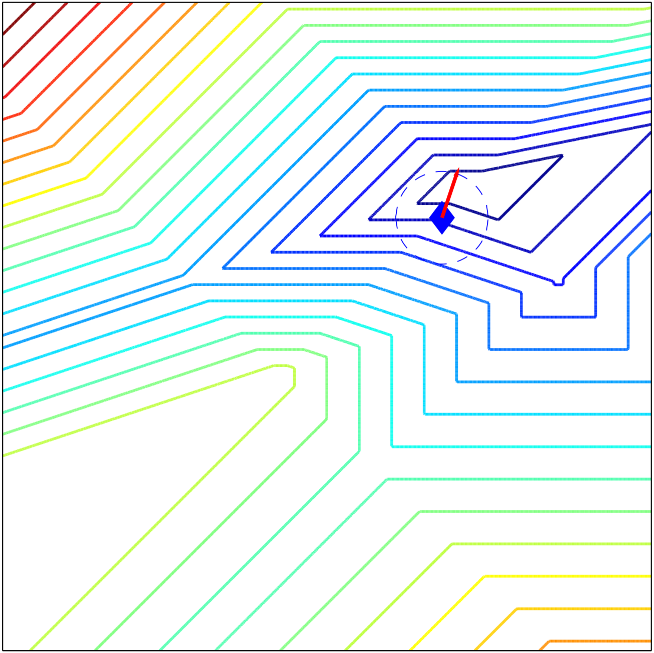


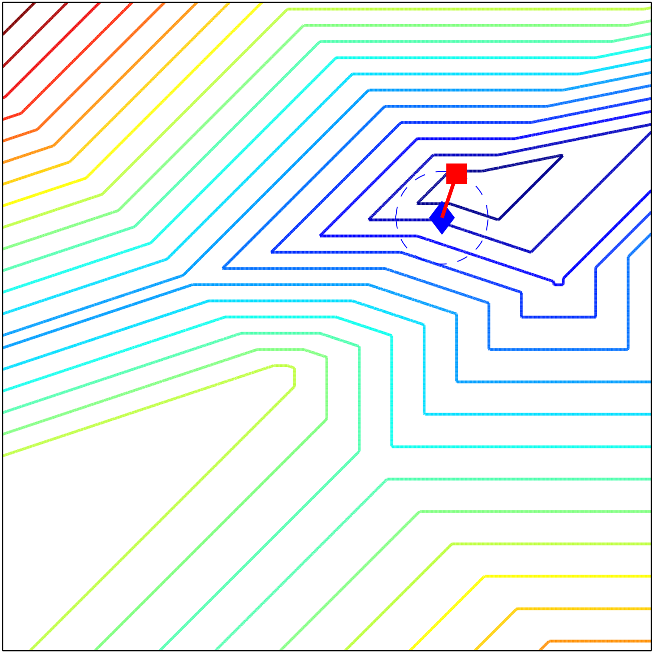


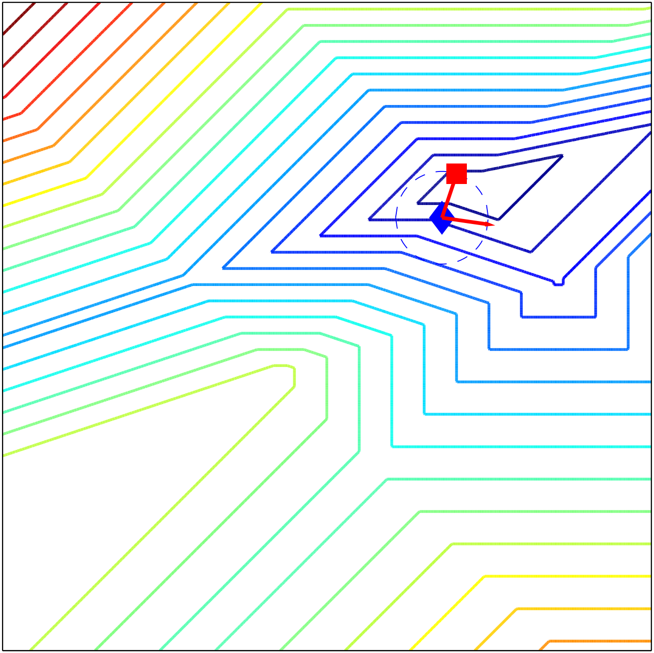


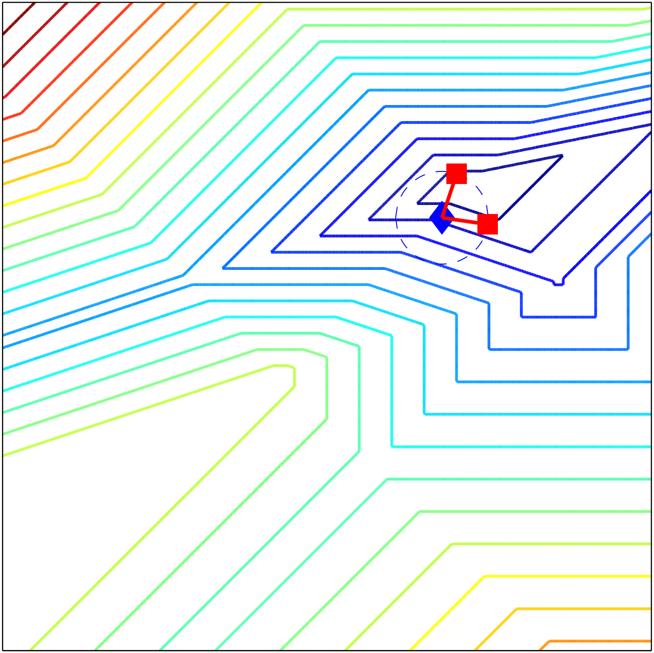


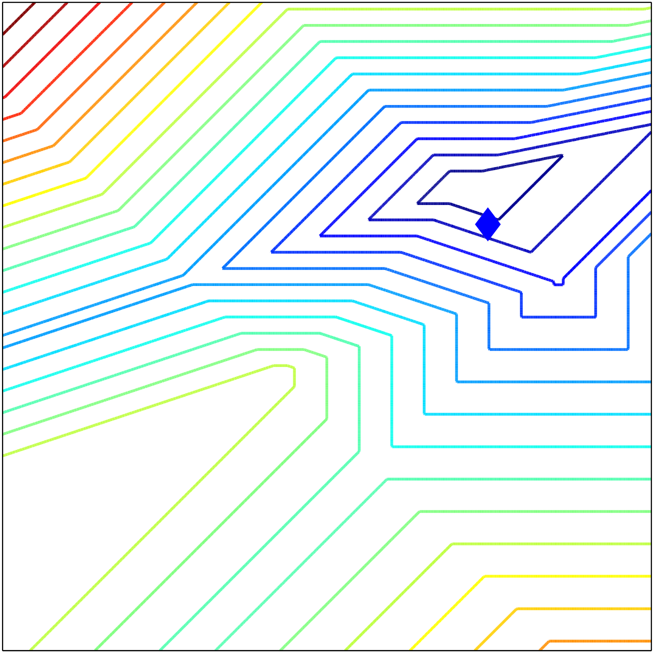


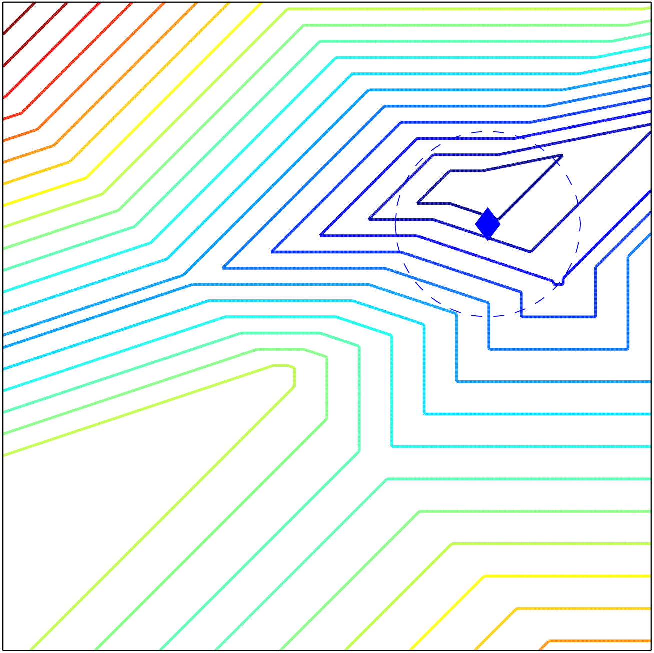


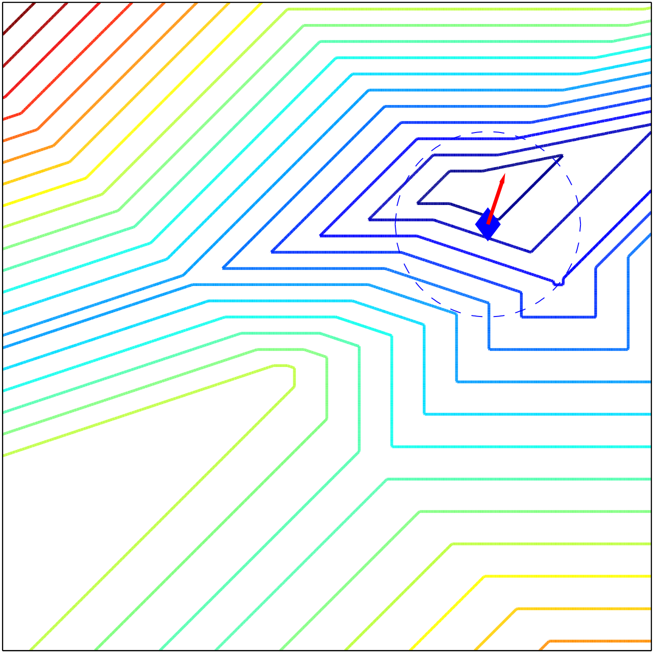


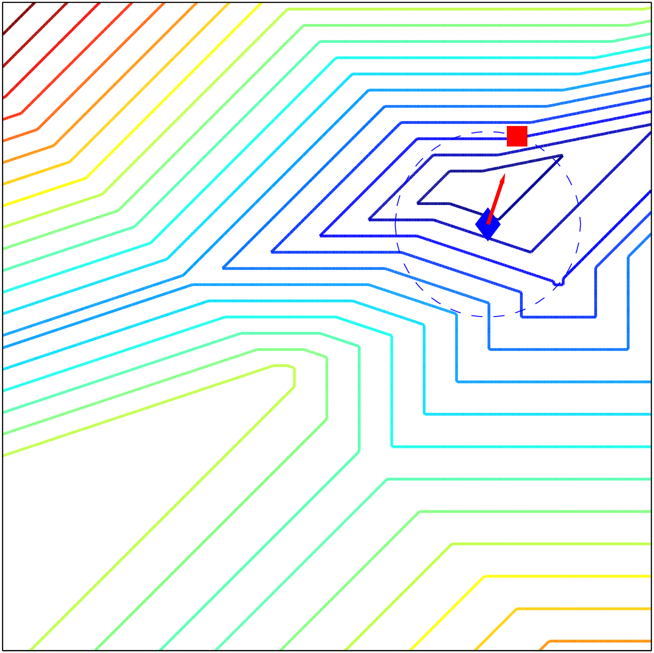


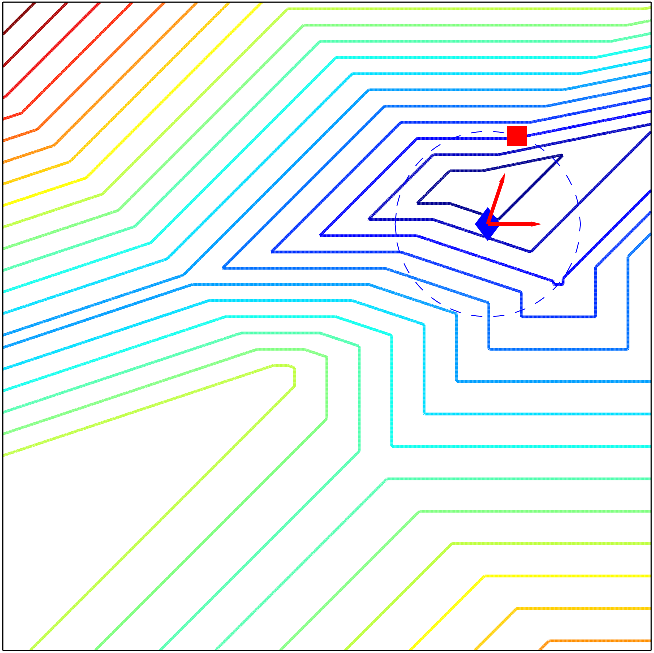


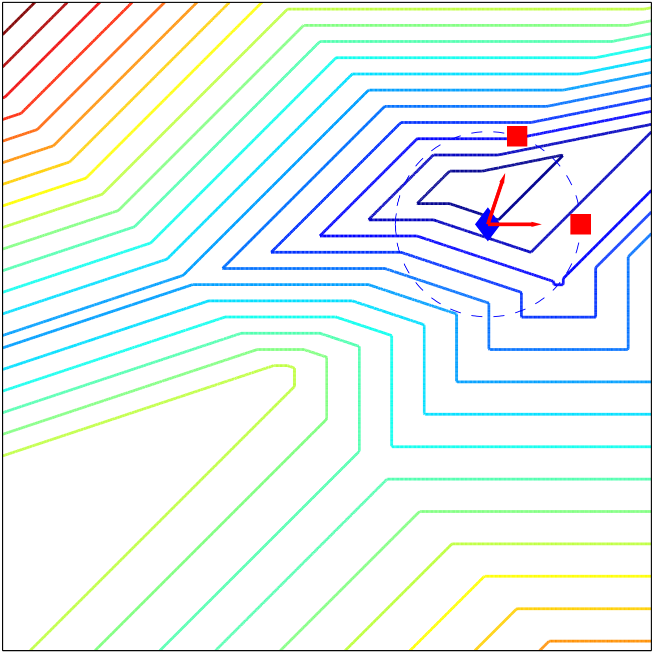


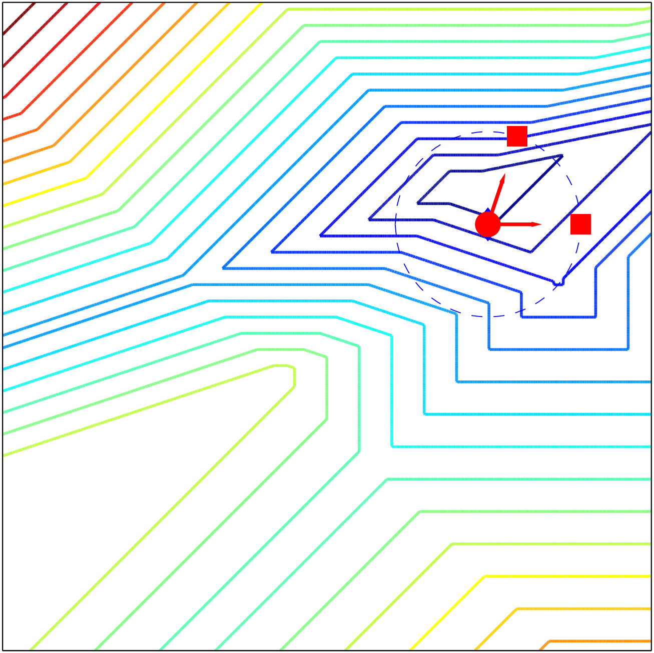


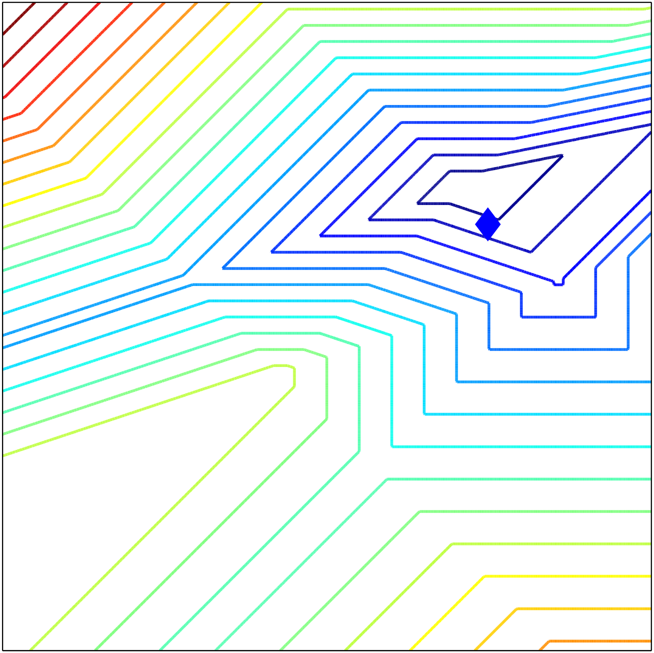


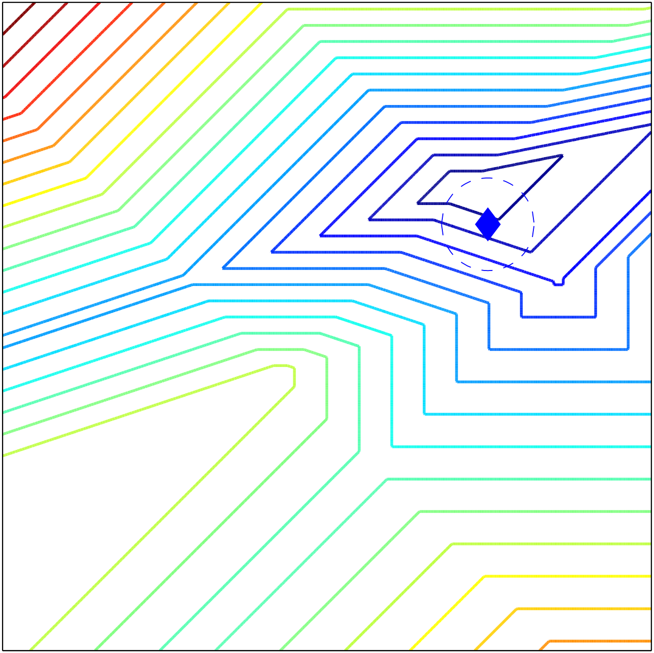


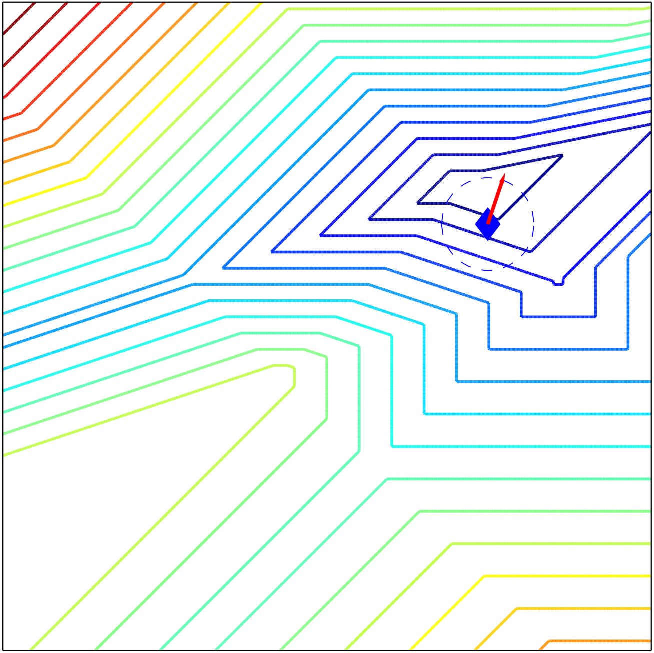


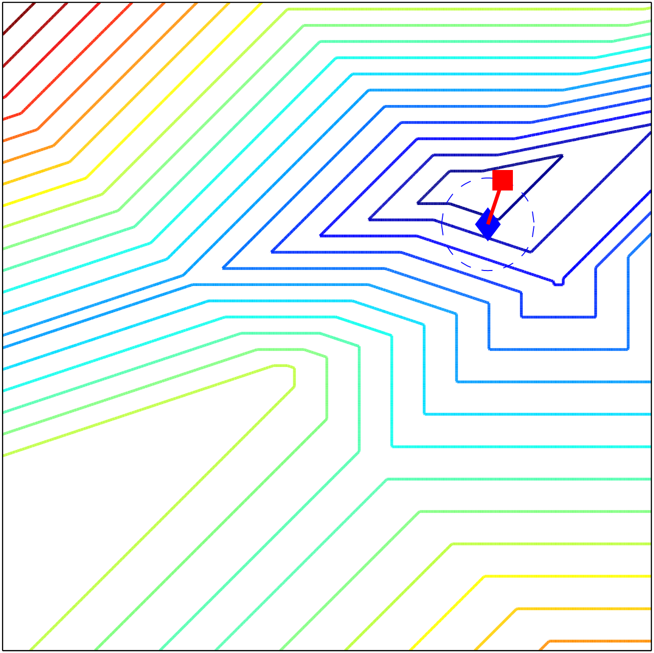


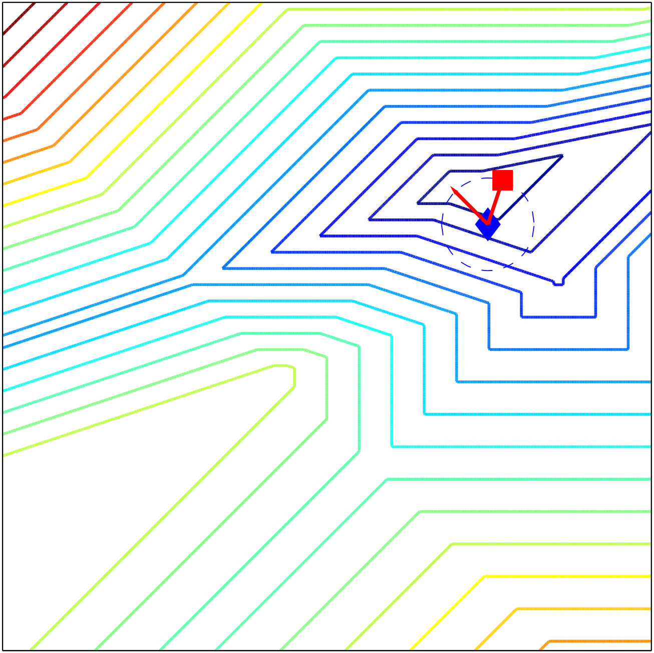


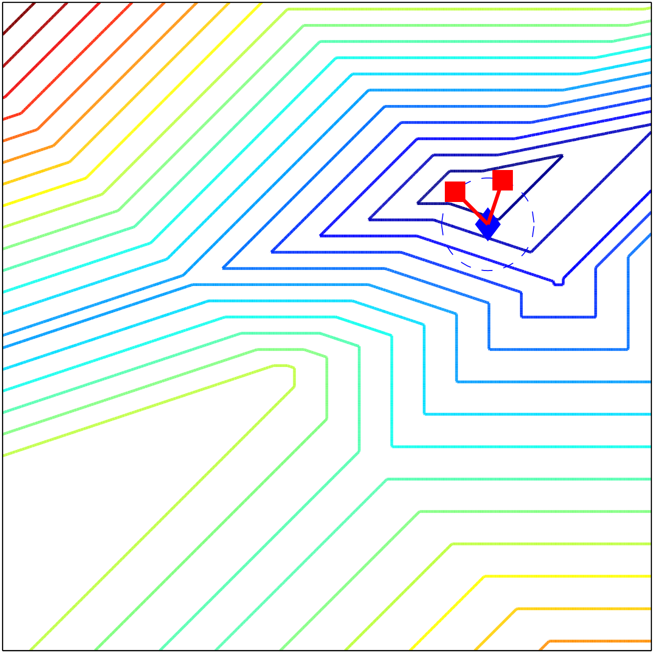


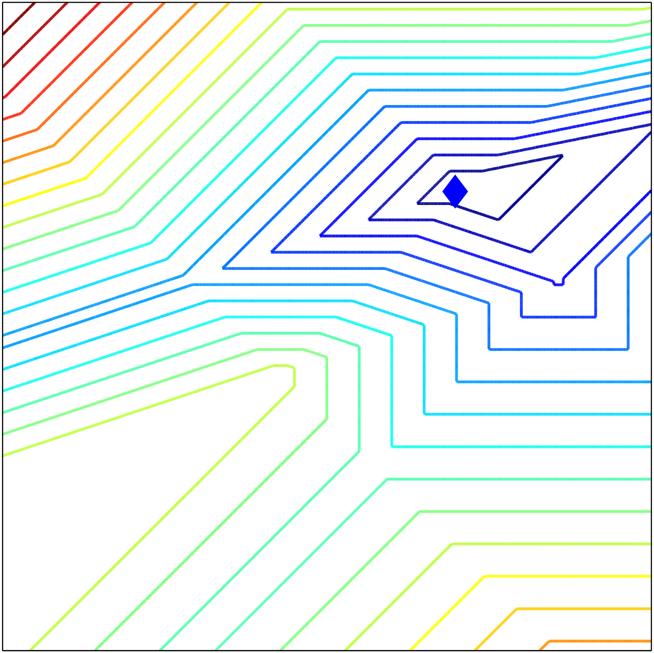












Generator set

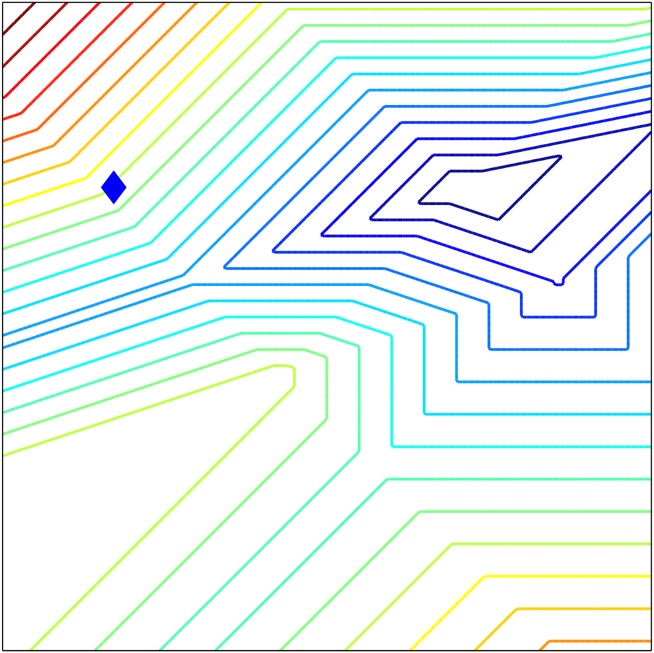
At some iterate x^k,

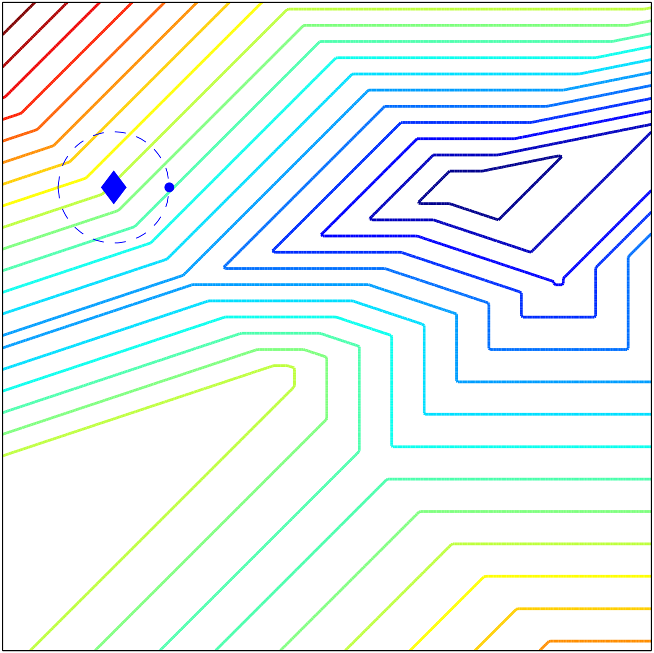
$$\mathfrak{G}^k \triangleq \bigcup_{i \in I_h(F(x^k))} \left\{ \nabla\psi(x^k) + \nabla M(x^k) a_i \right\}$$

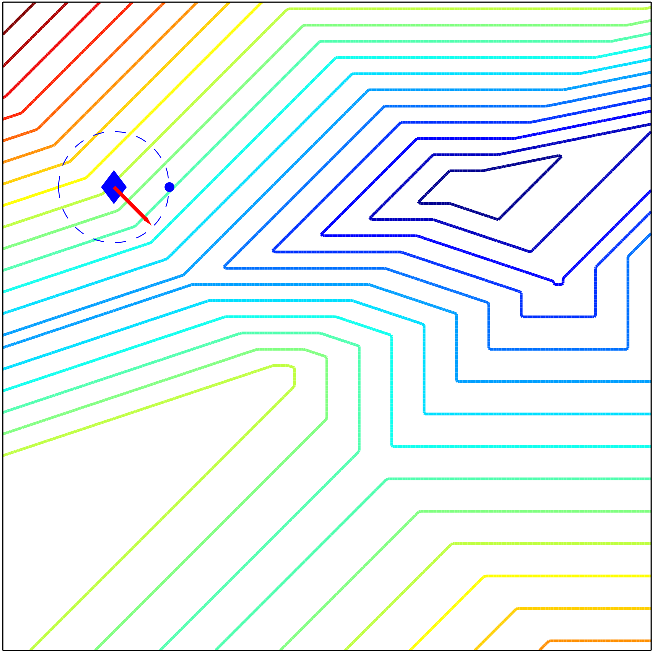
where $I_h(F(x^k))$ is the set of essentially active indices.

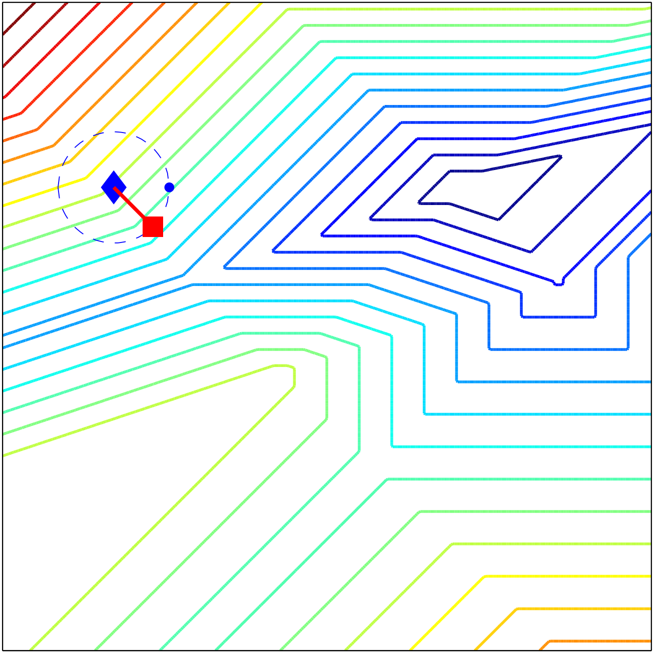
Or, given a set of points $Y = \left\{x^k, y^2, \ldots, y^p\right\} \subset \mathcal{B}(x^k, \Delta_k)$,

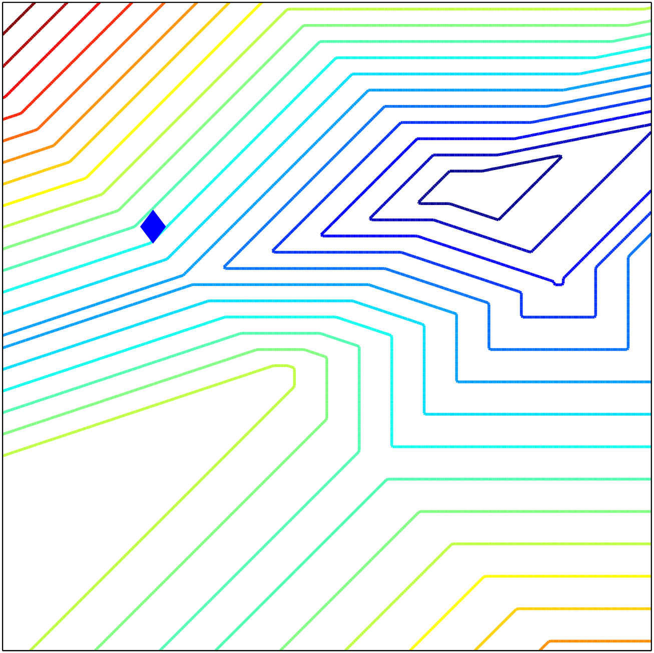
$$\mathfrak{G}^k \triangleq \bigcup_{y \in Y} \bigcup_{i \in I_h(F(y))} \left\{ \nabla\psi(x^k) + \nabla M(x^k) a_i \right\}$$

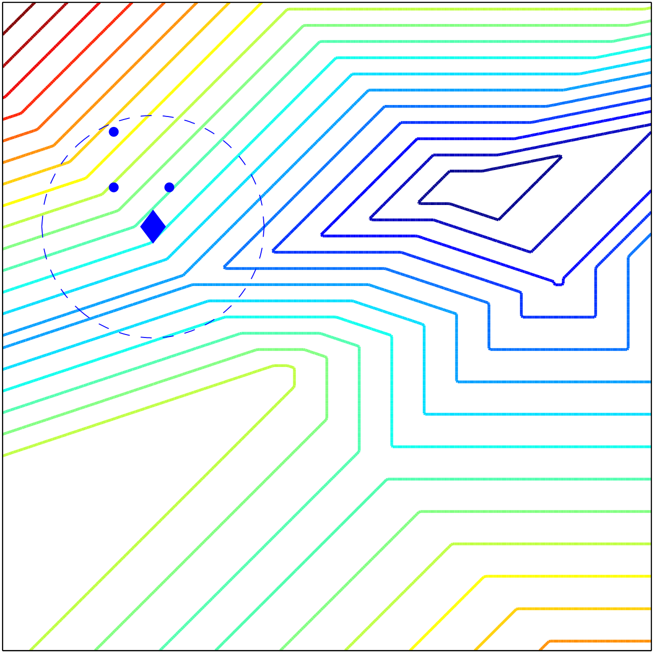


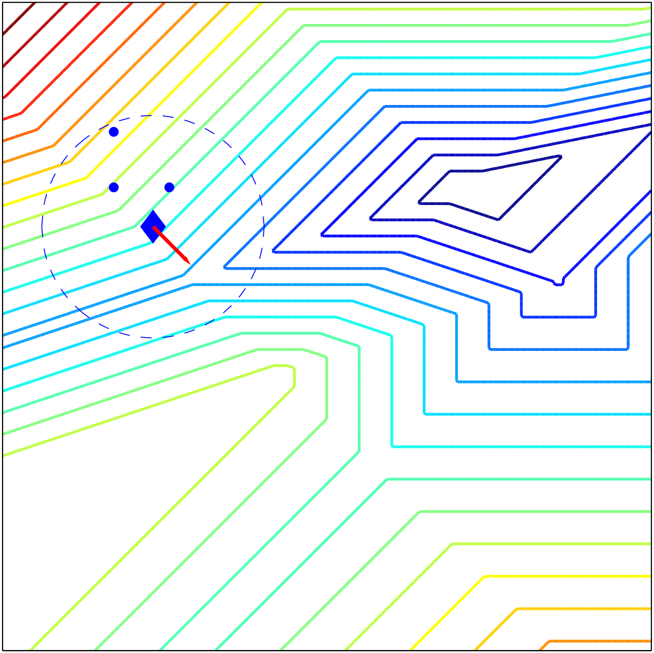


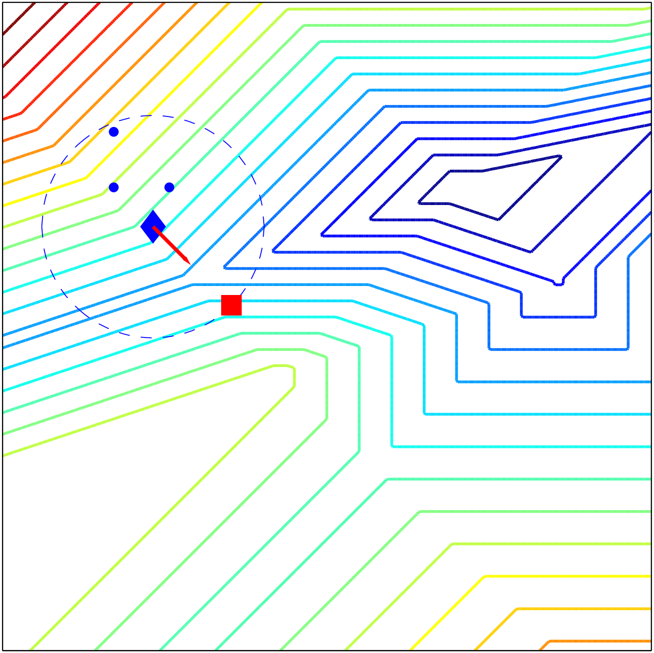


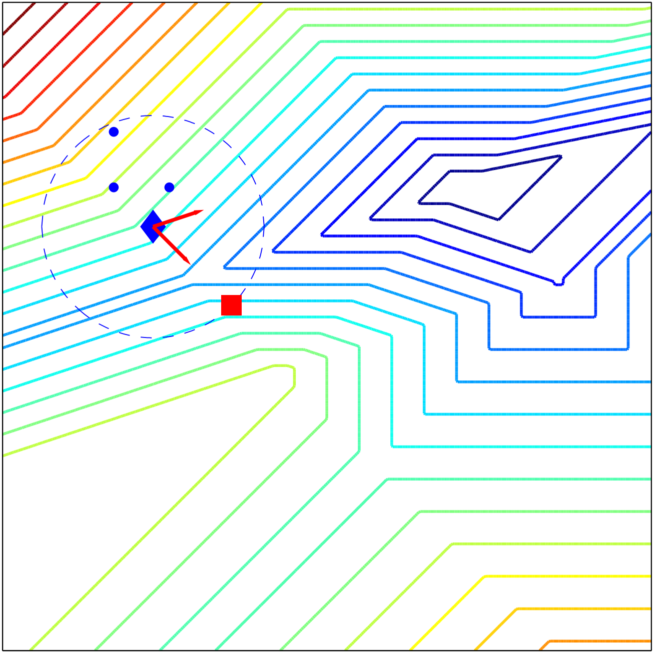


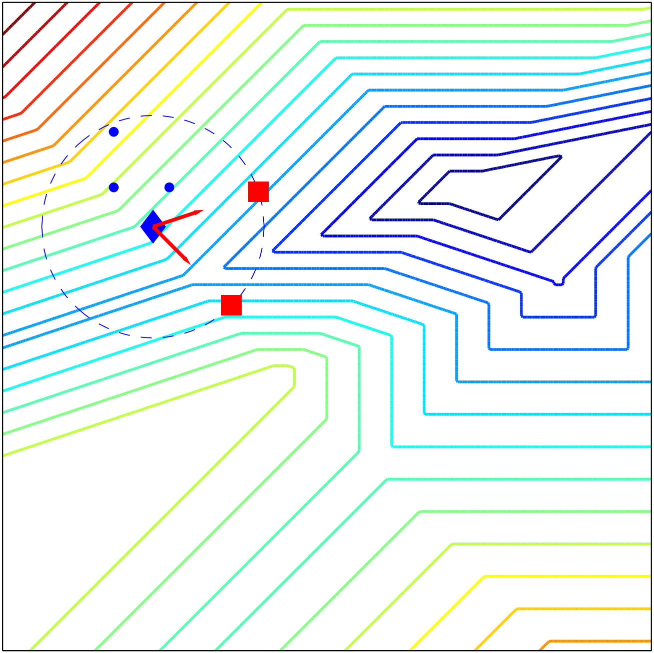


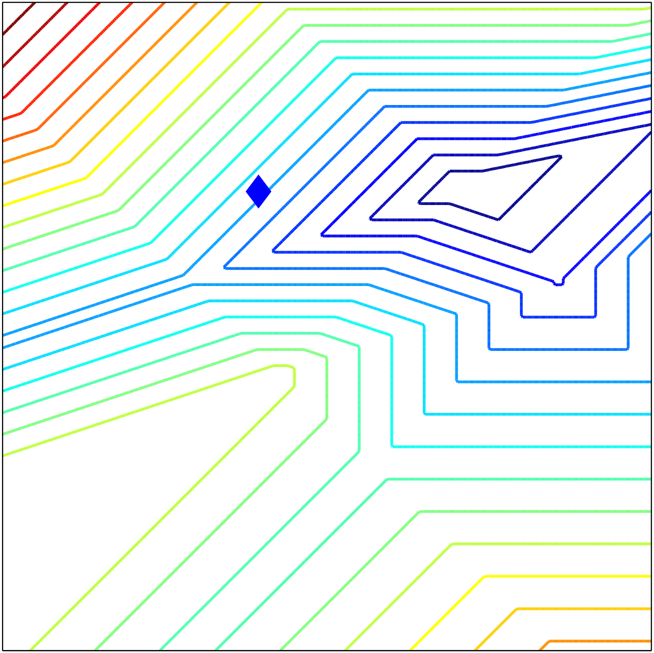


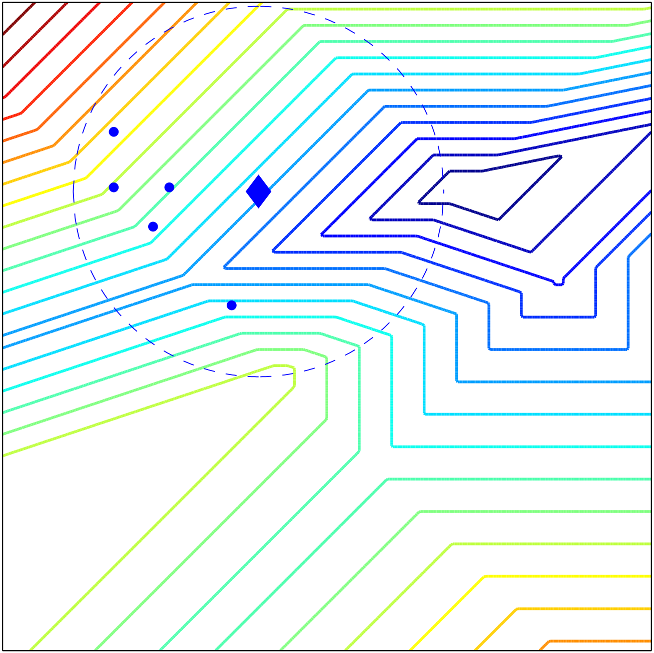


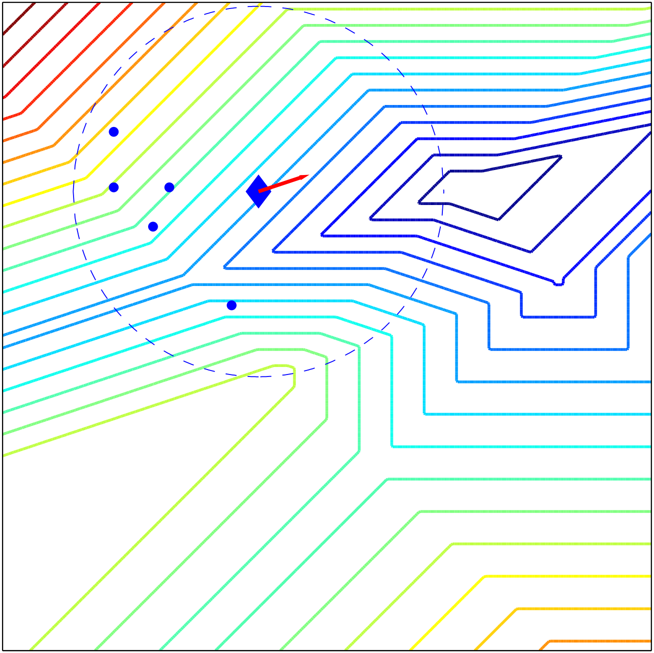


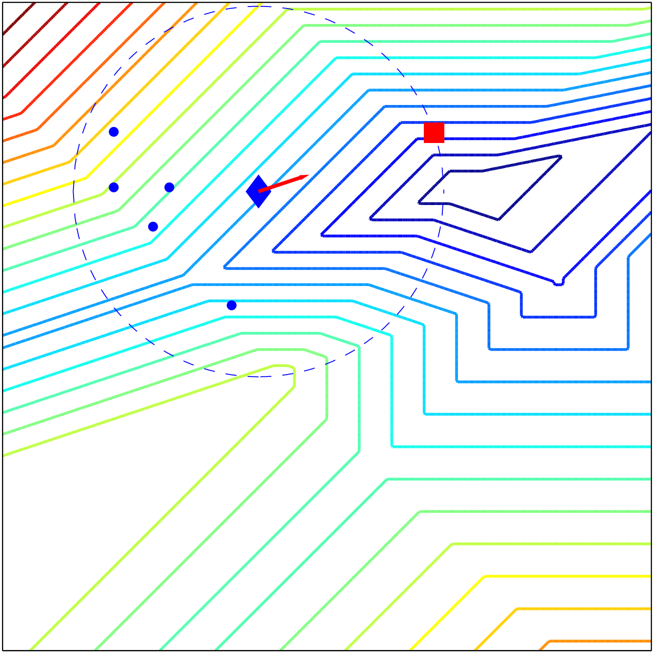


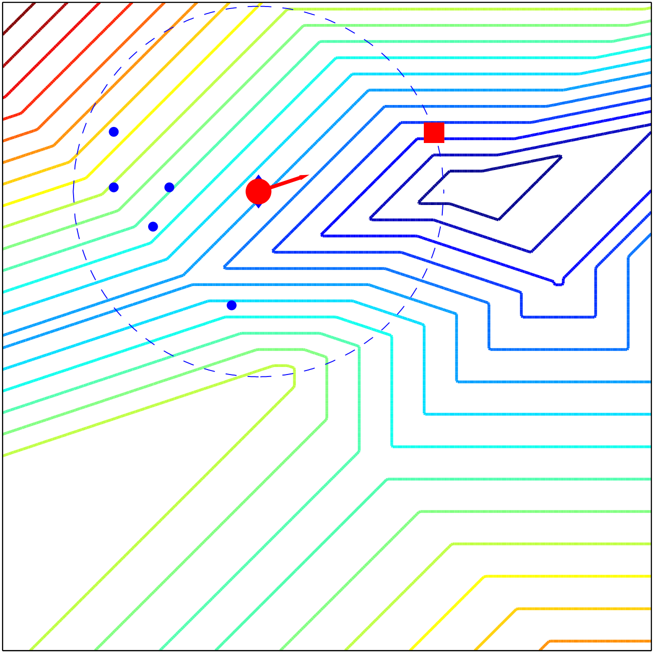


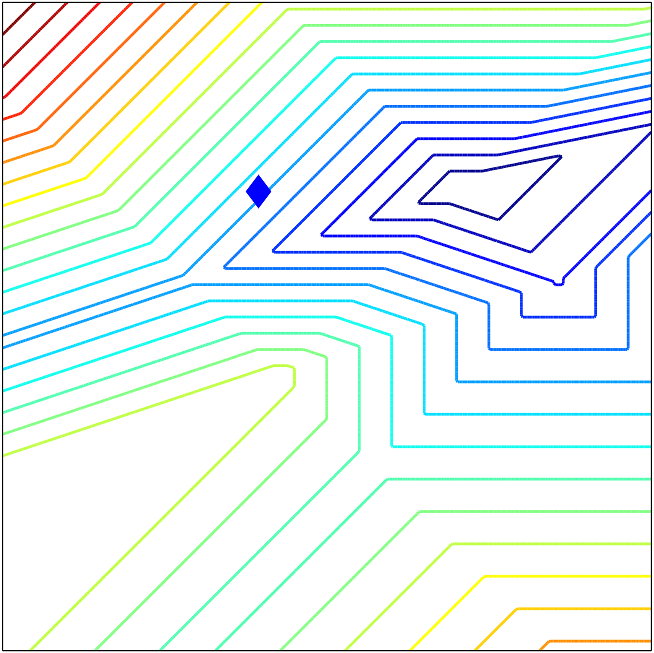


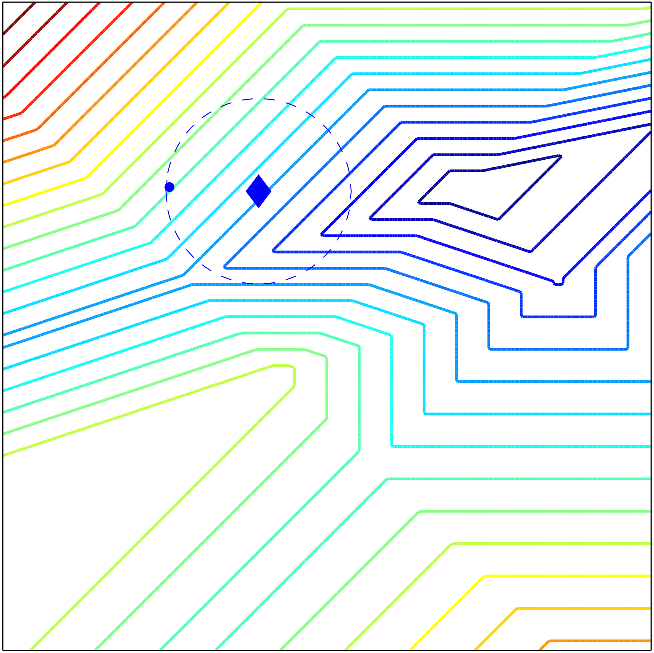


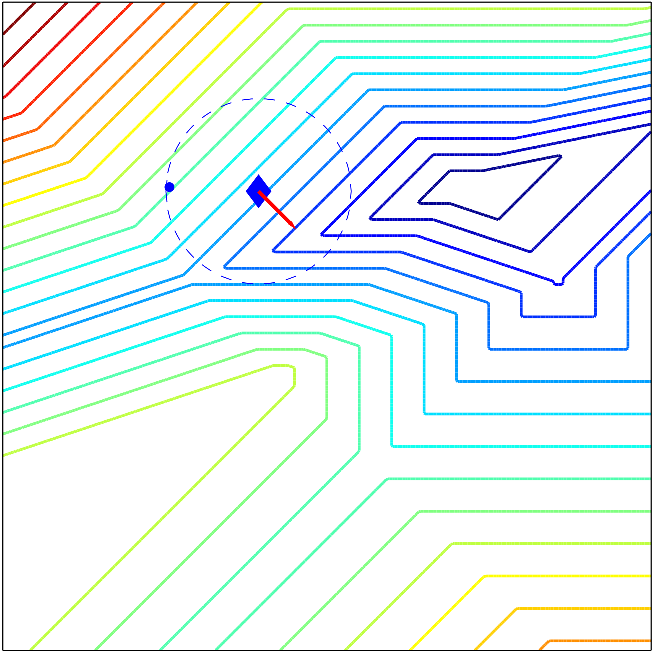


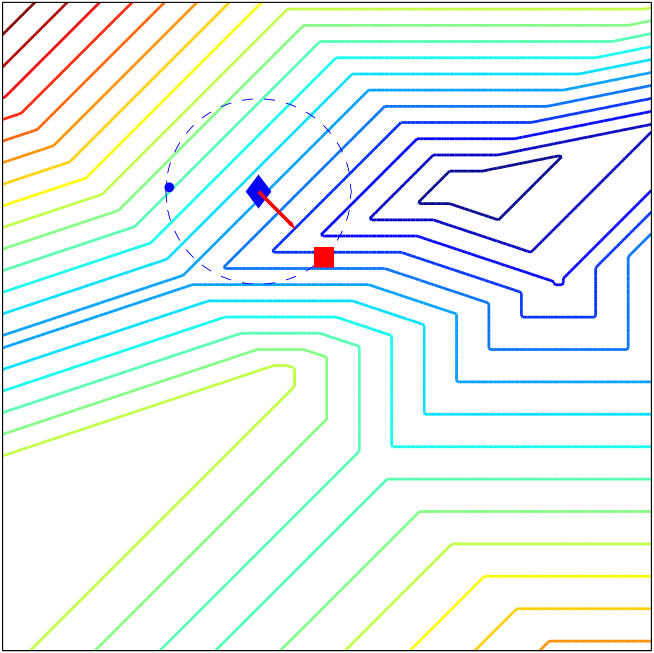


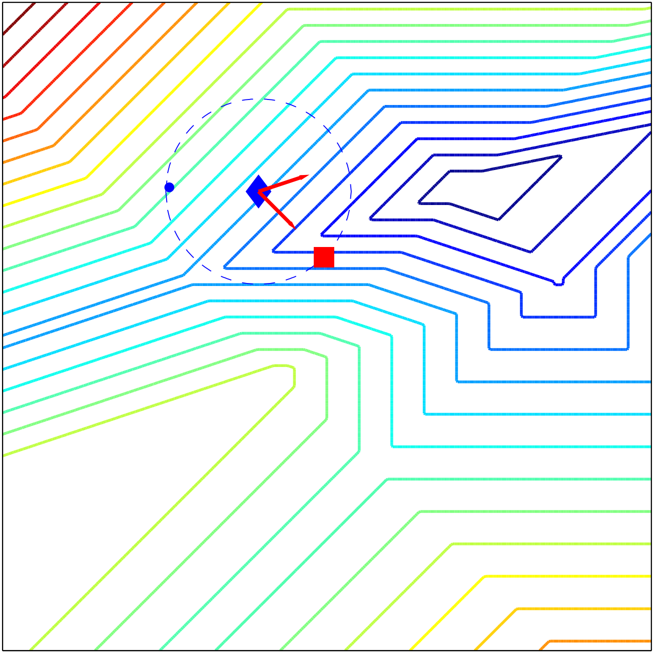


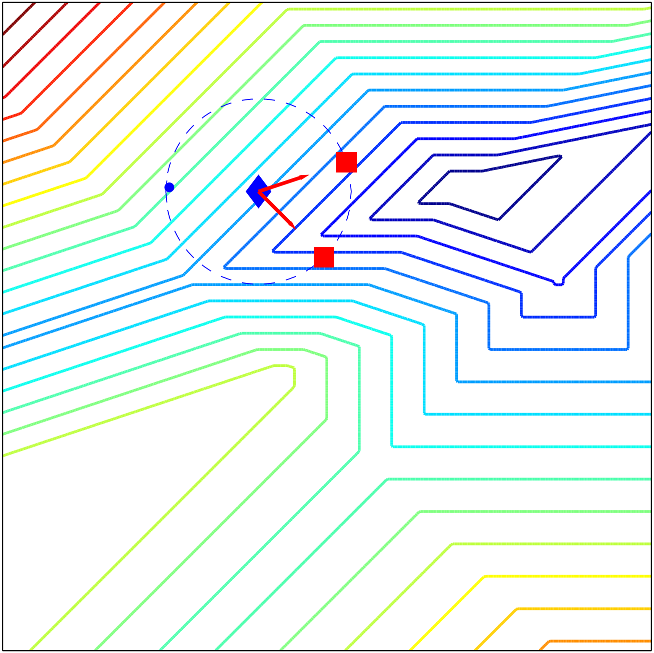


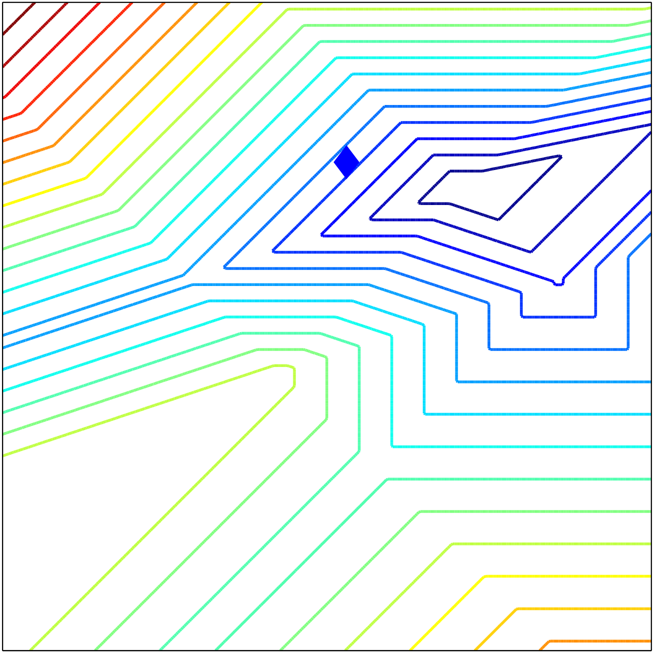


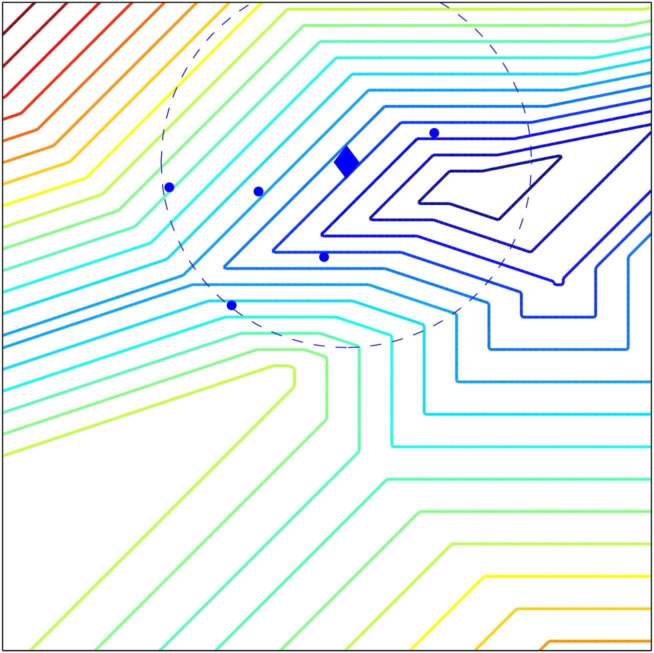


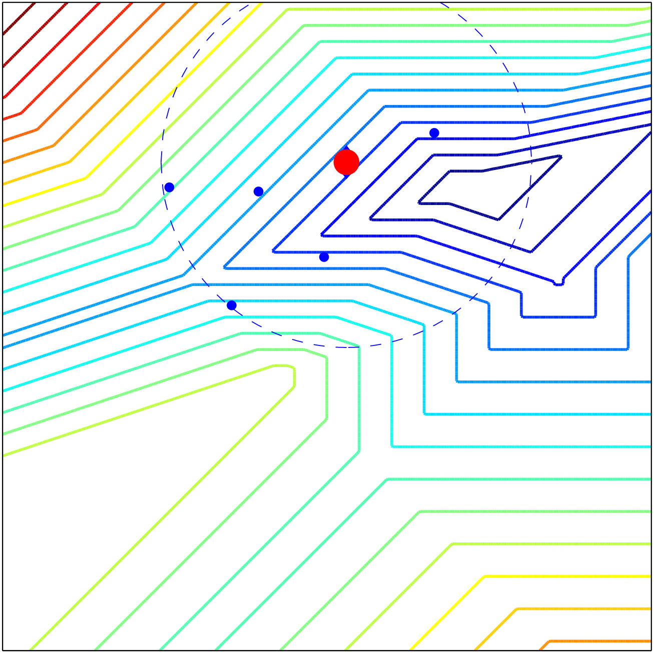


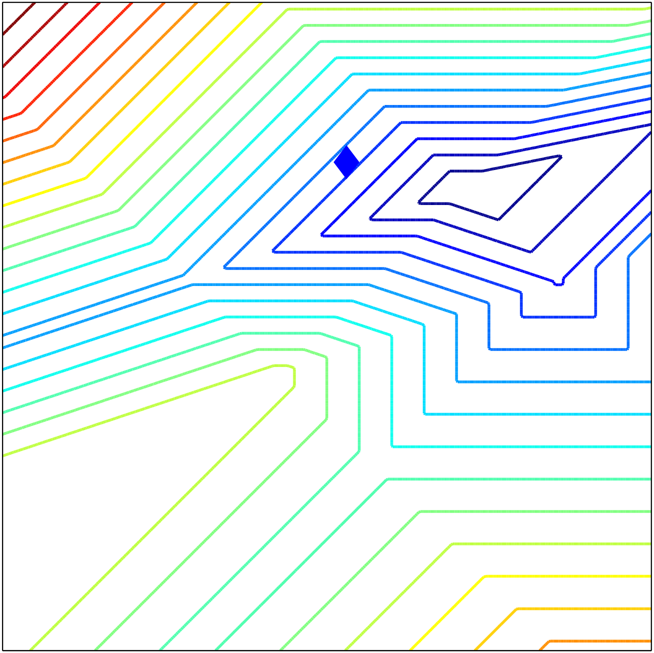


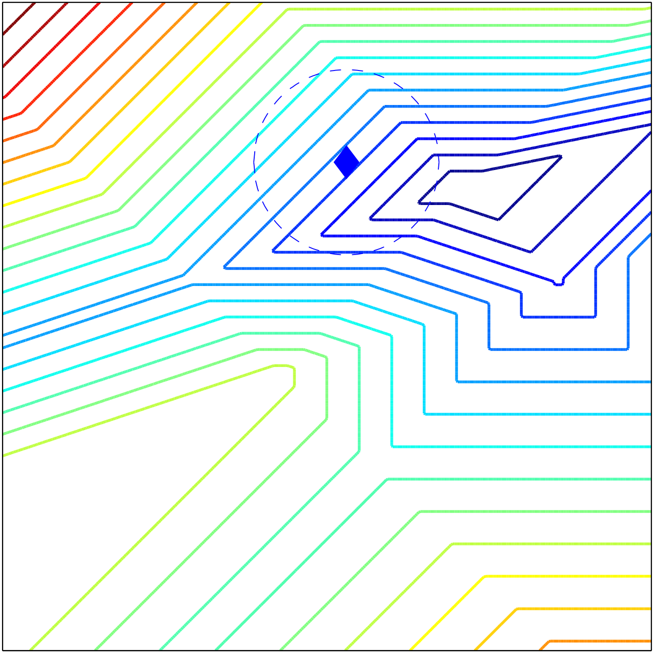


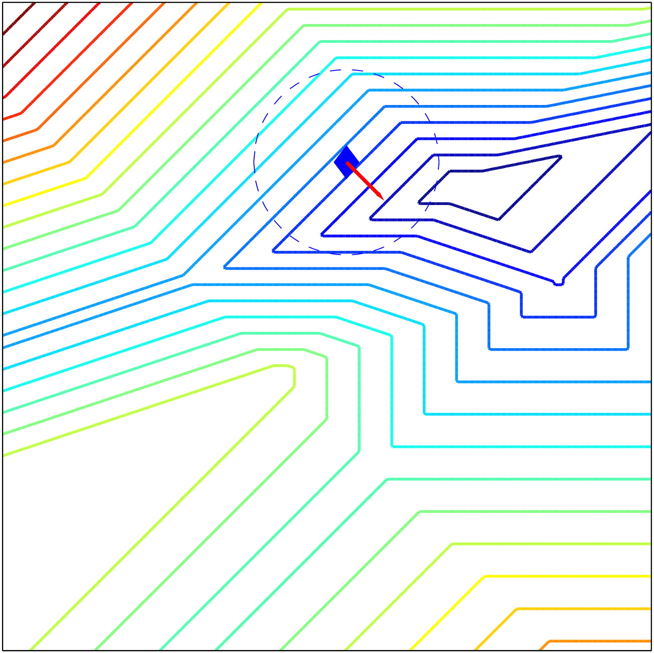


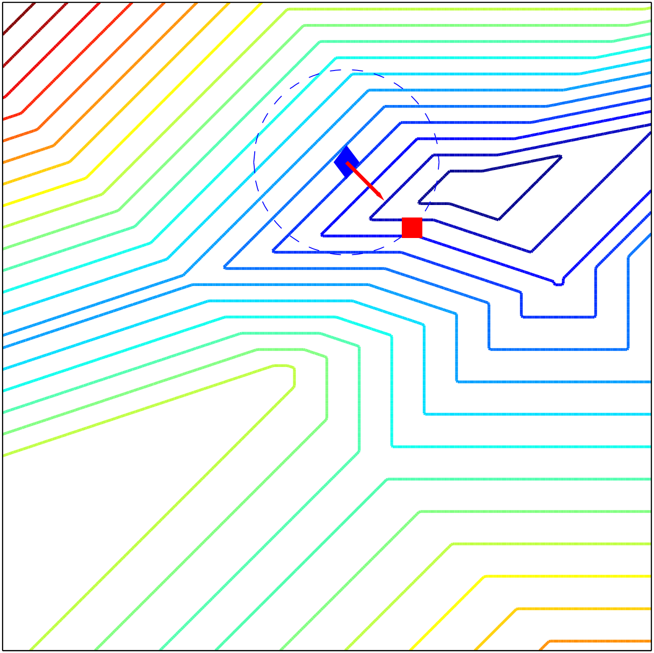


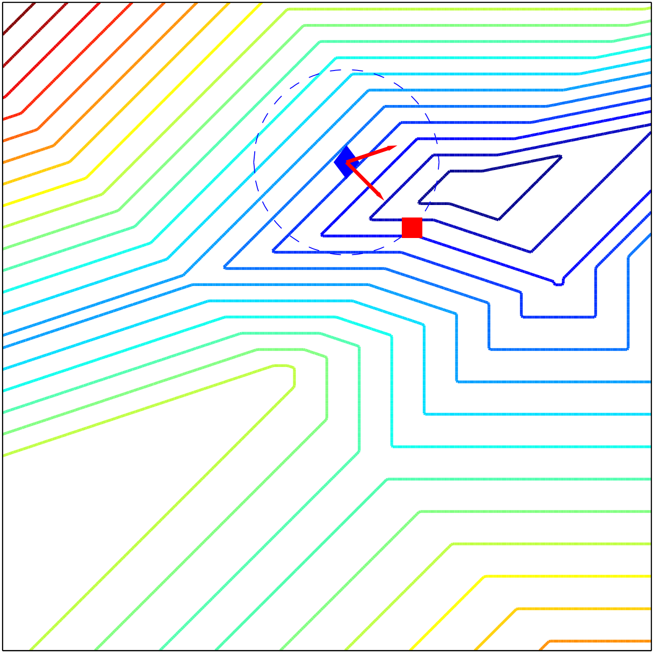


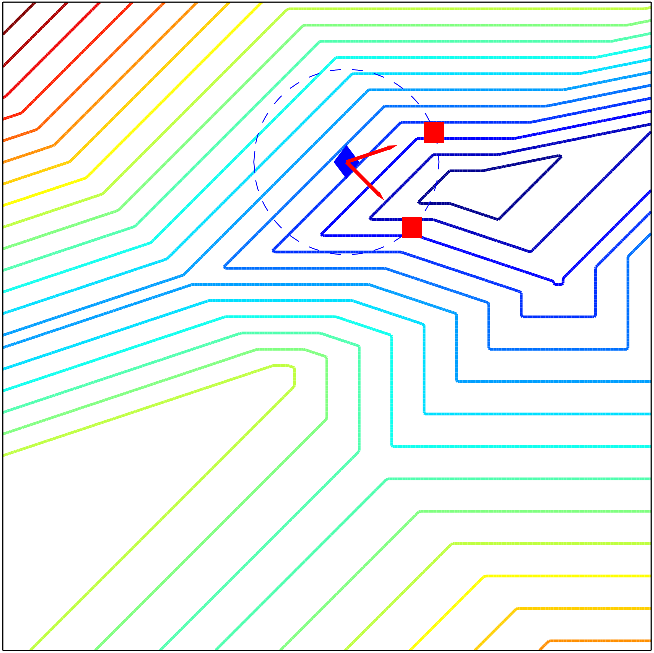


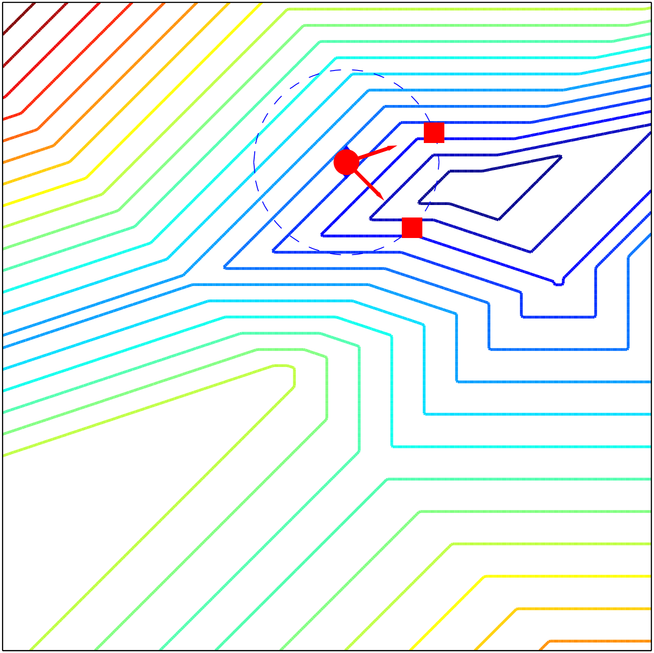


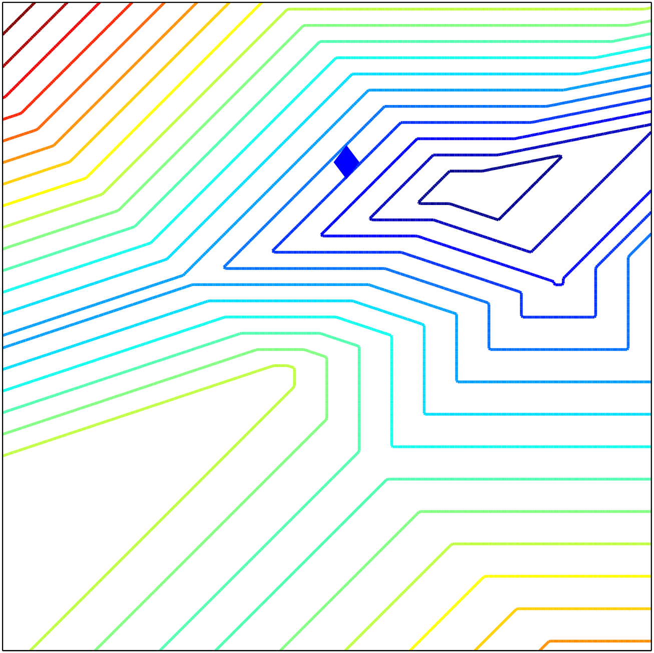


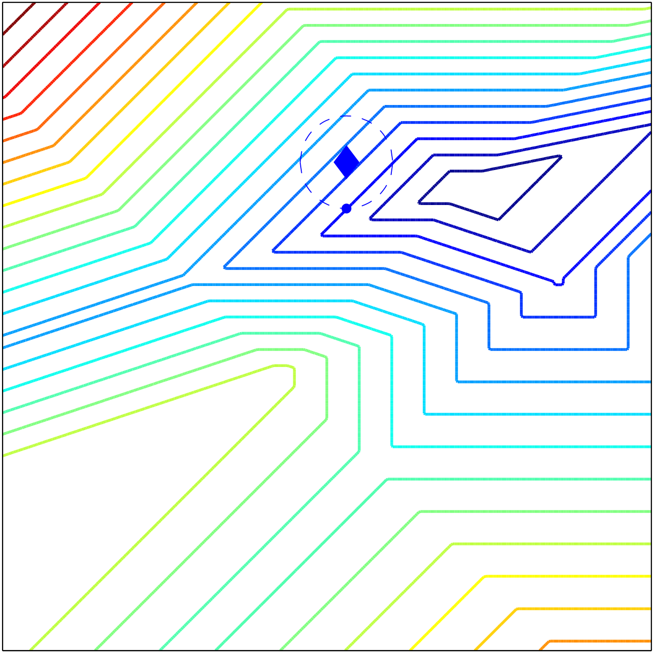


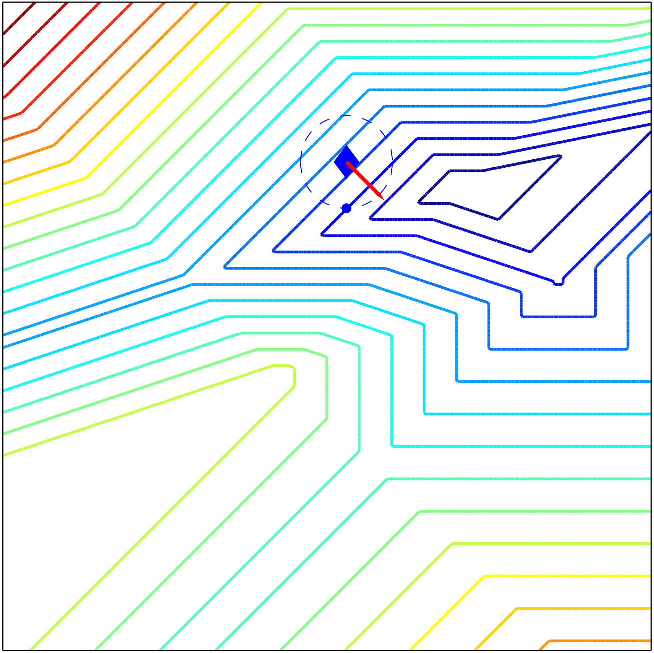


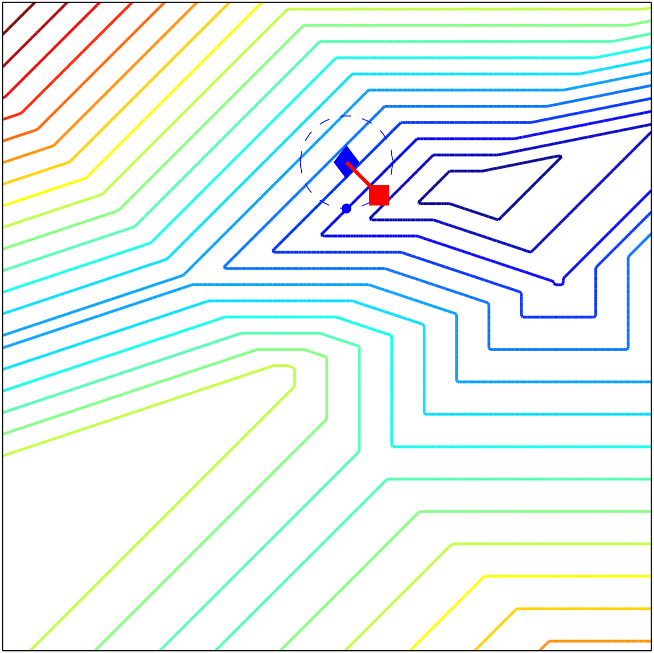


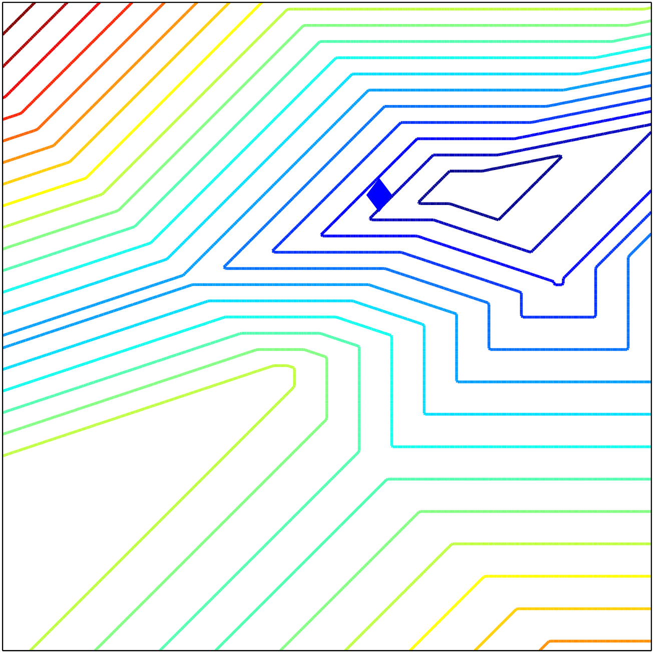


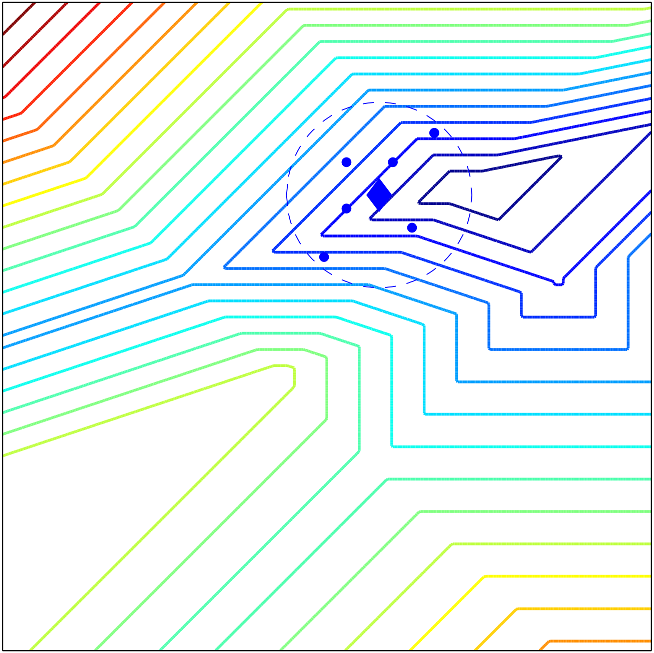


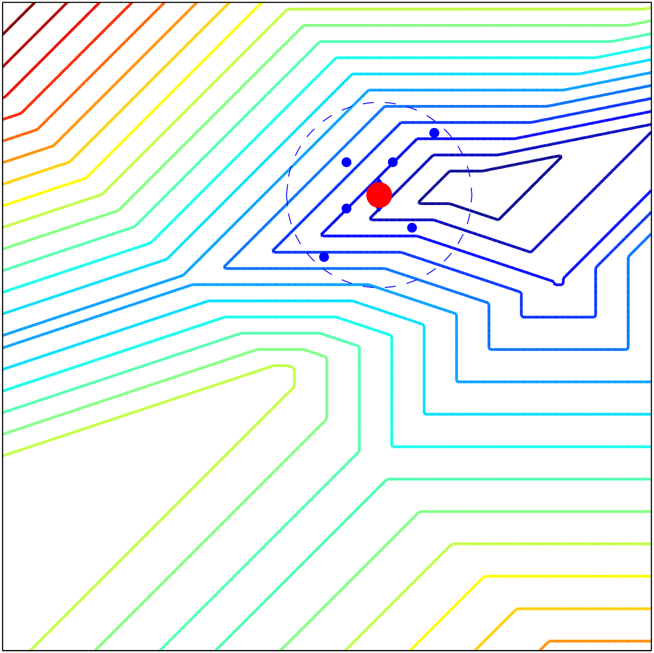


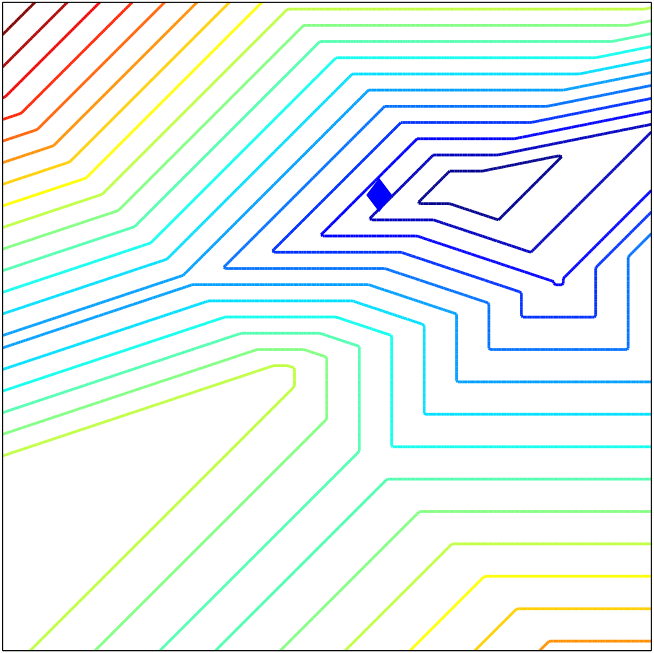


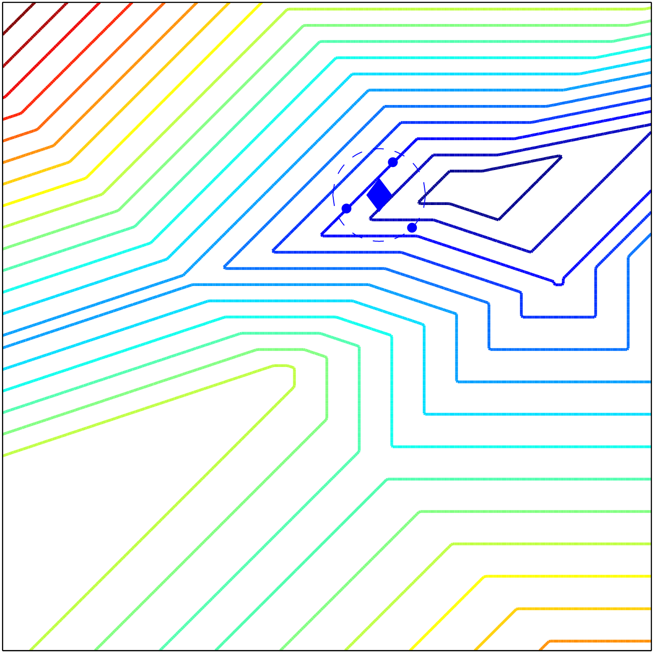


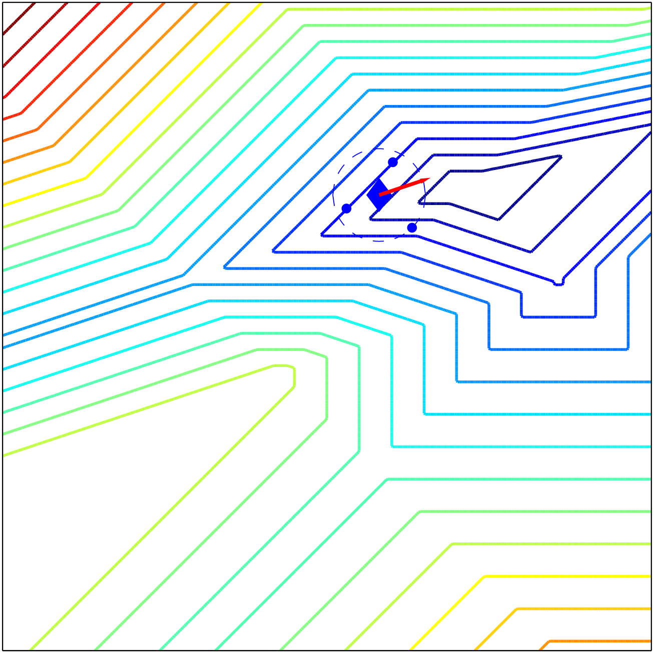


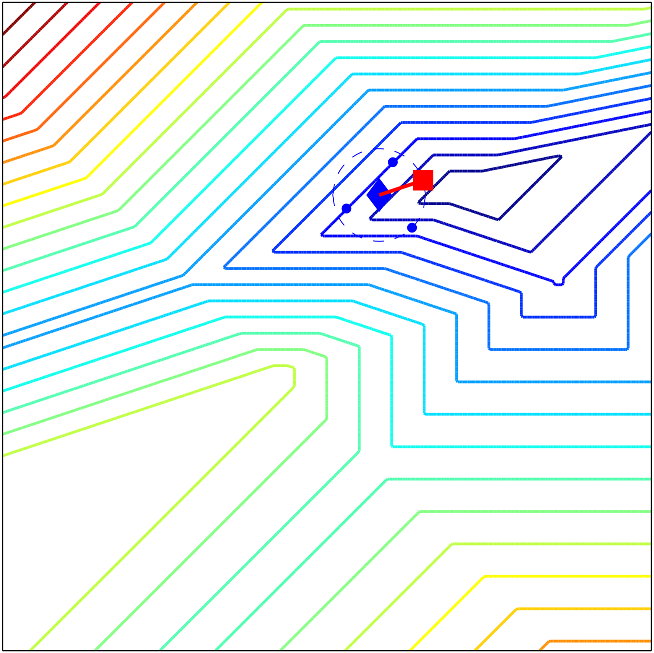


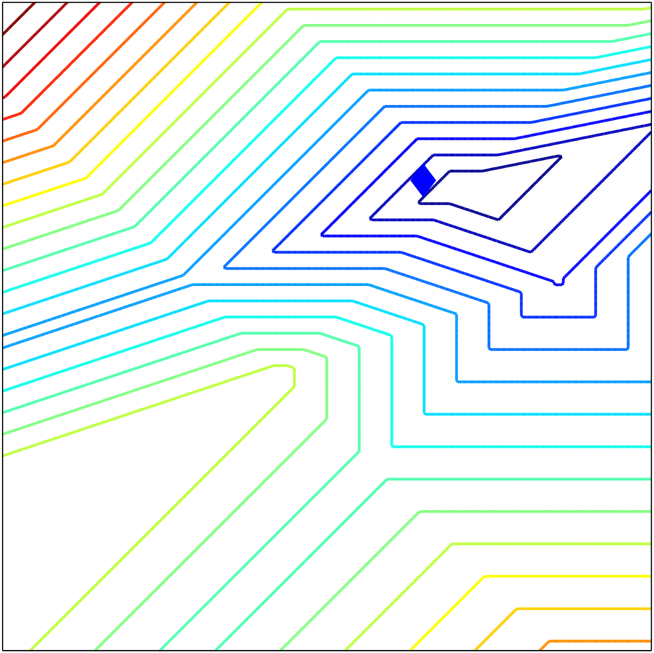


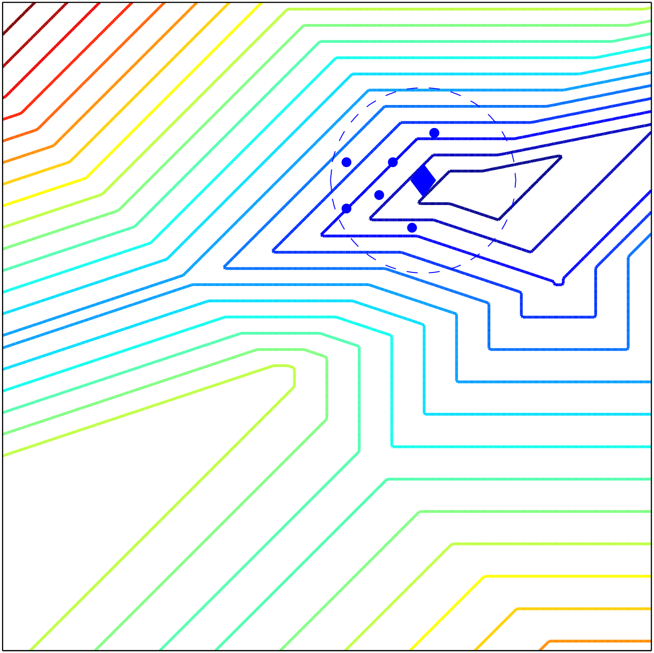


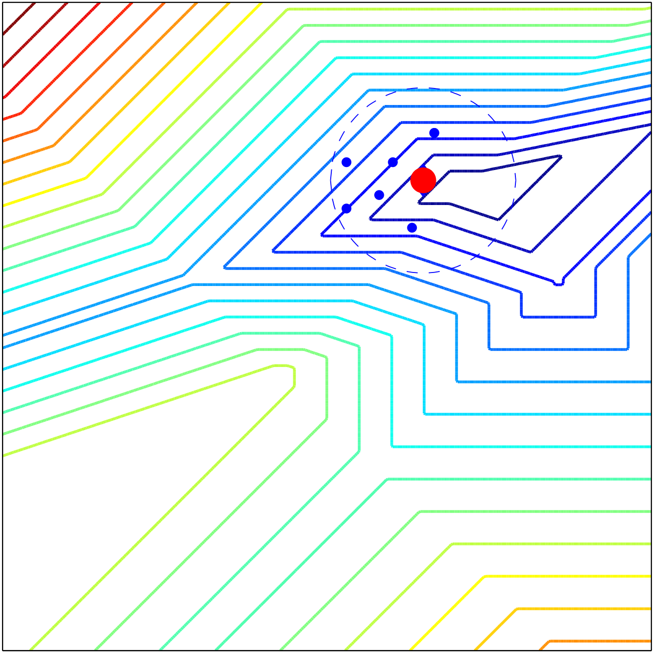


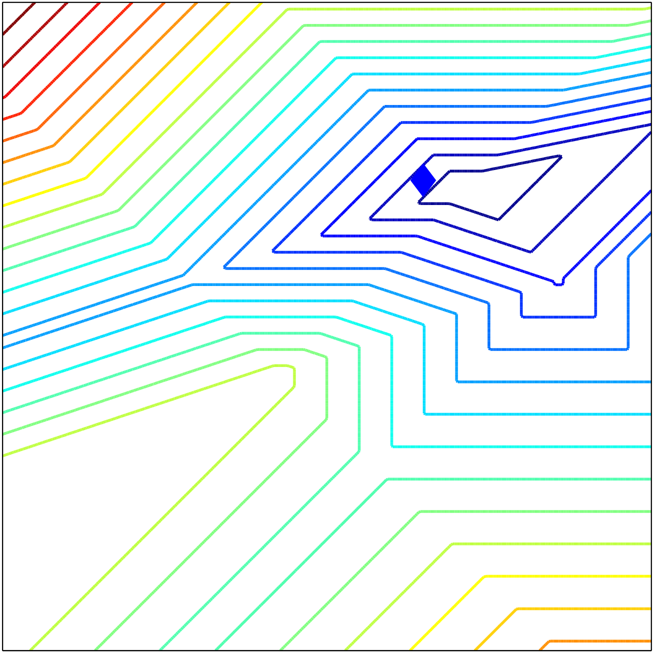


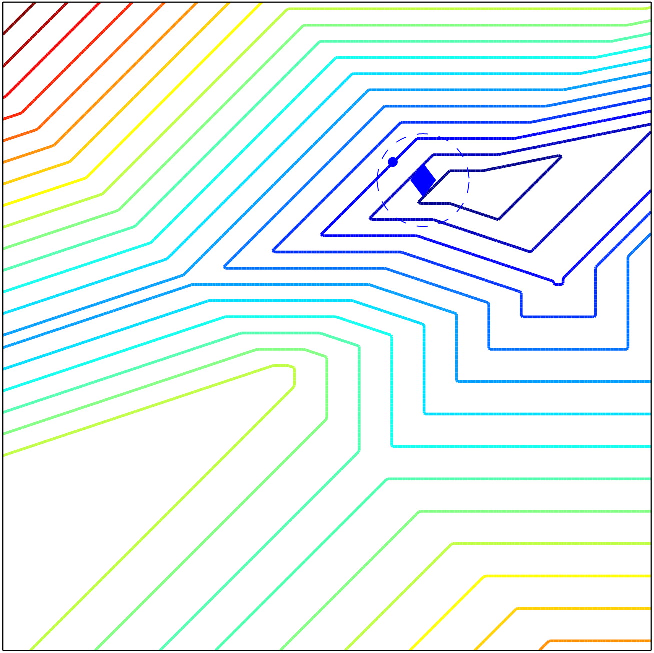


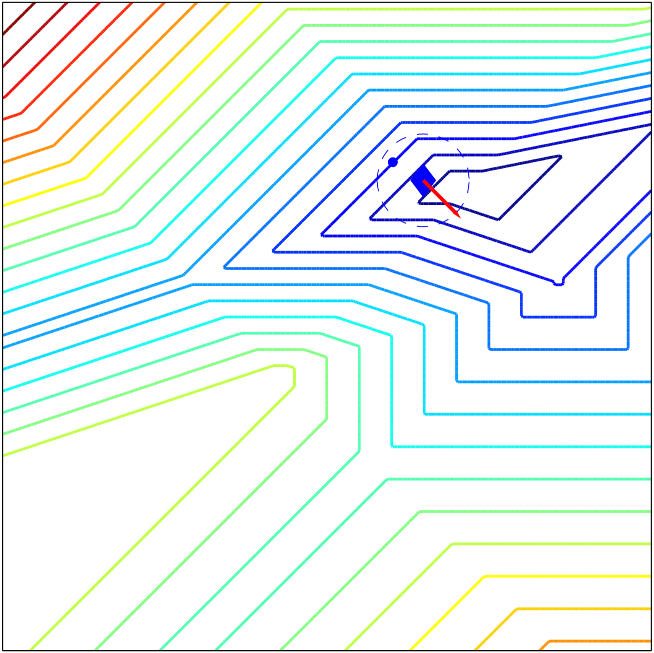


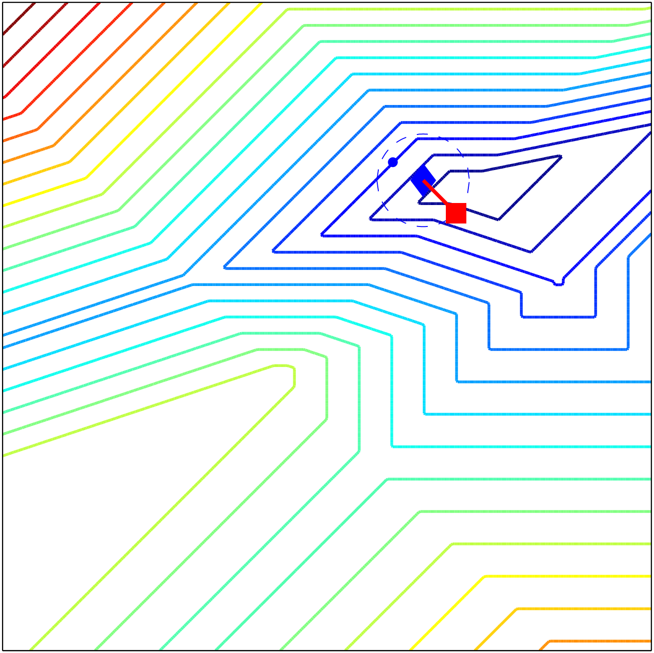


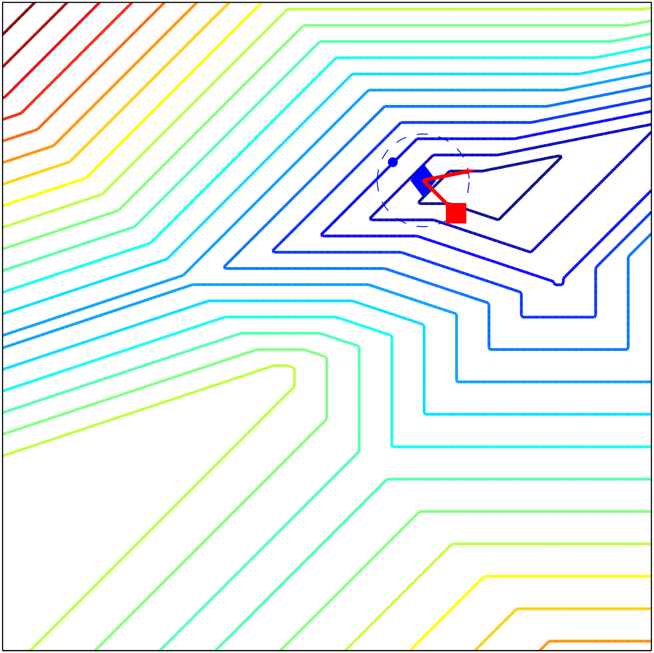


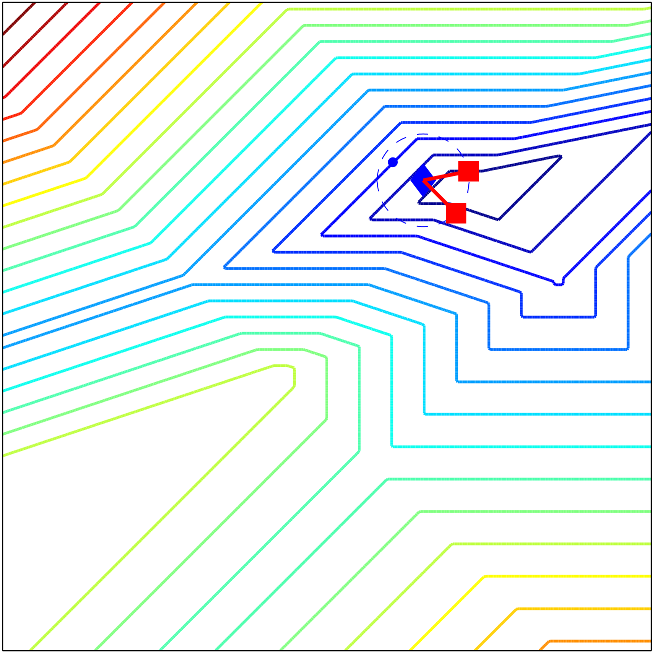


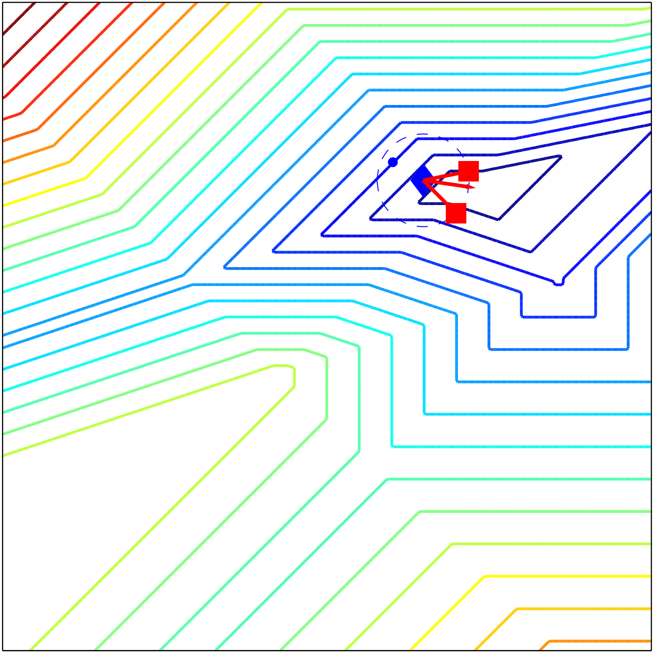


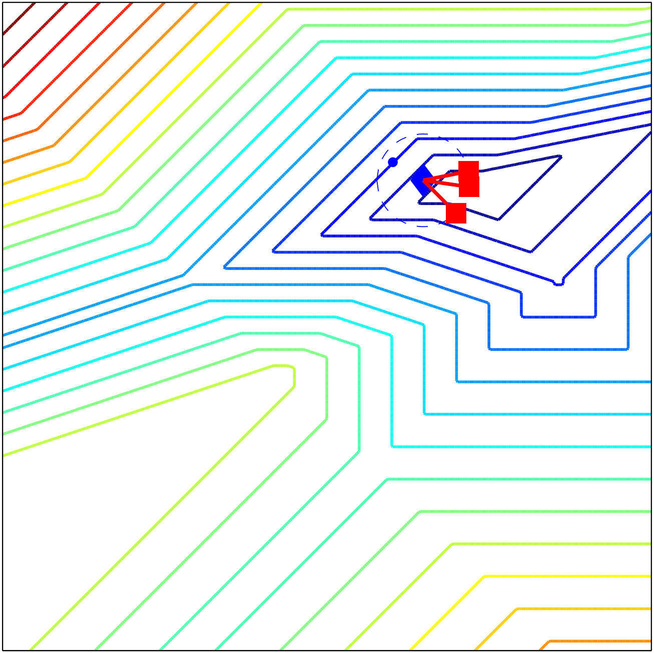


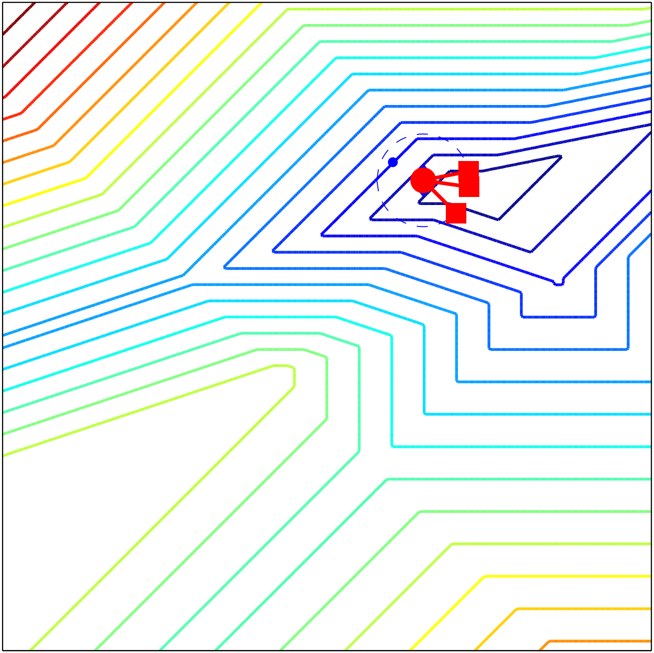


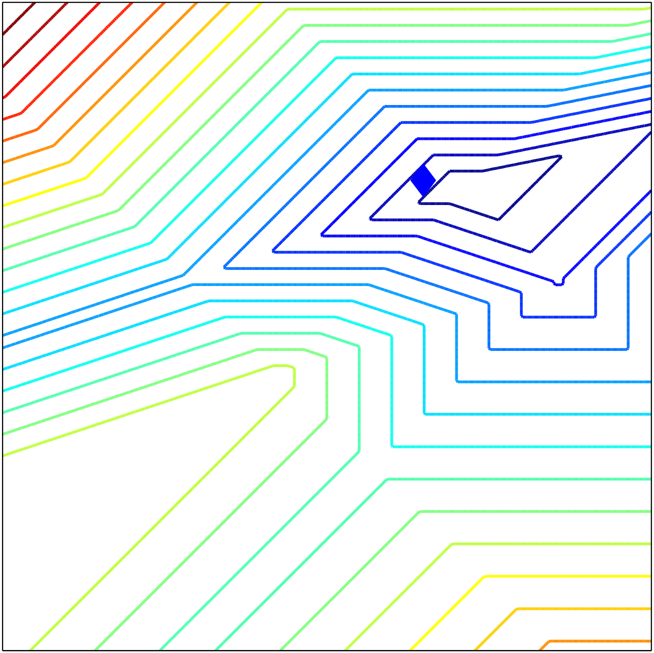


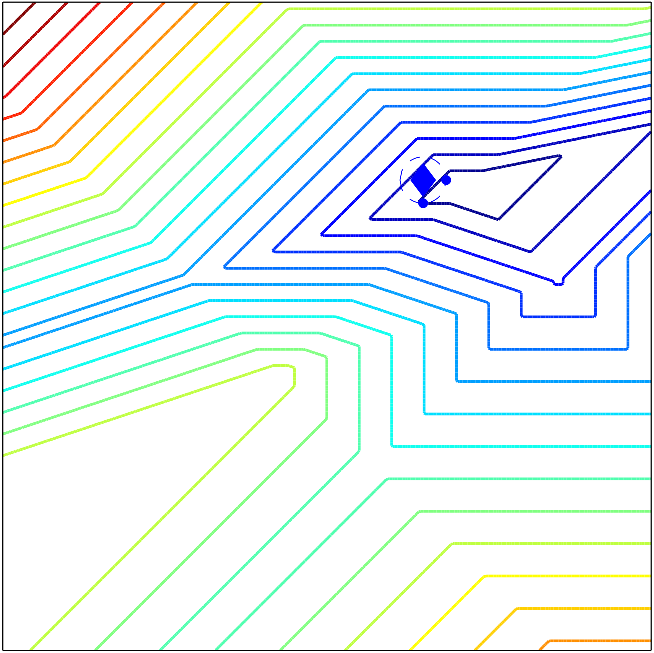


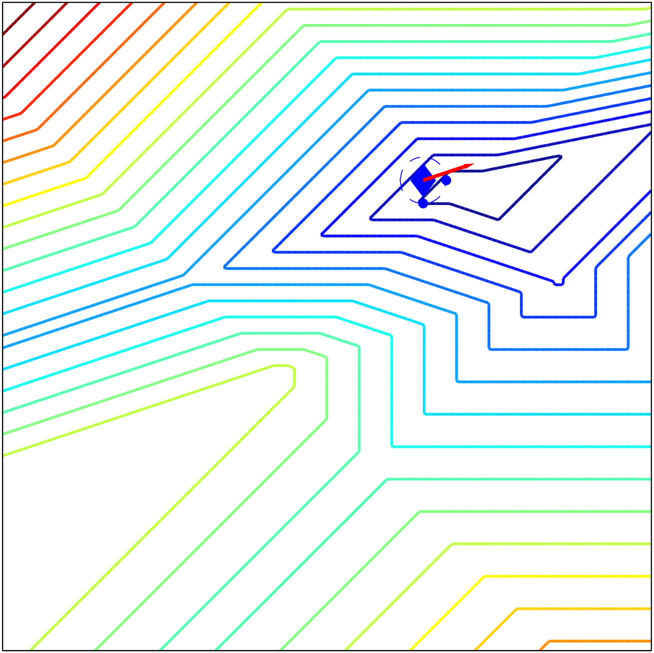


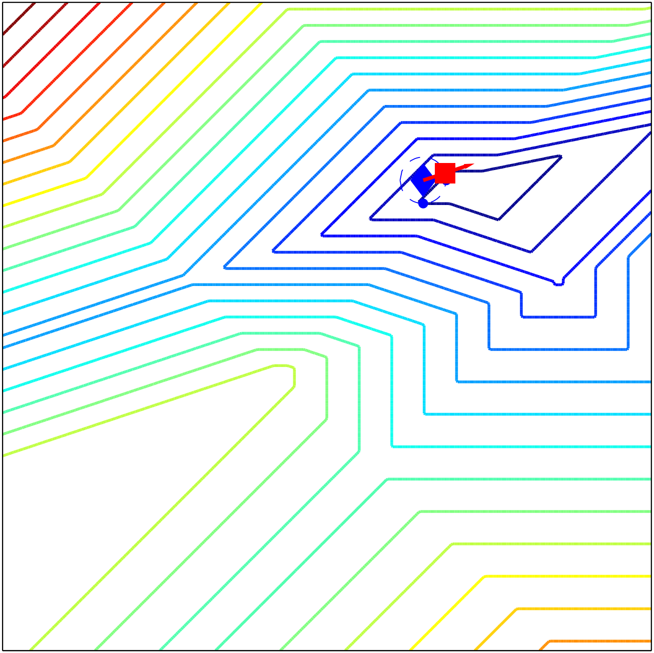


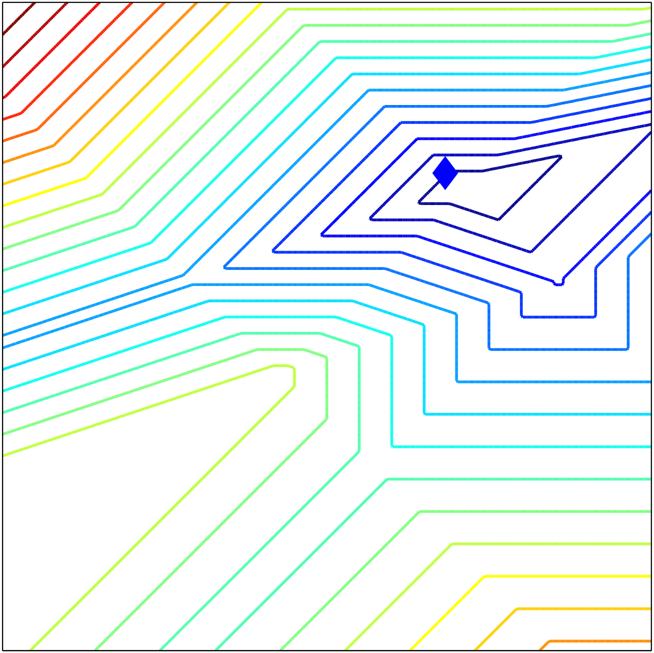


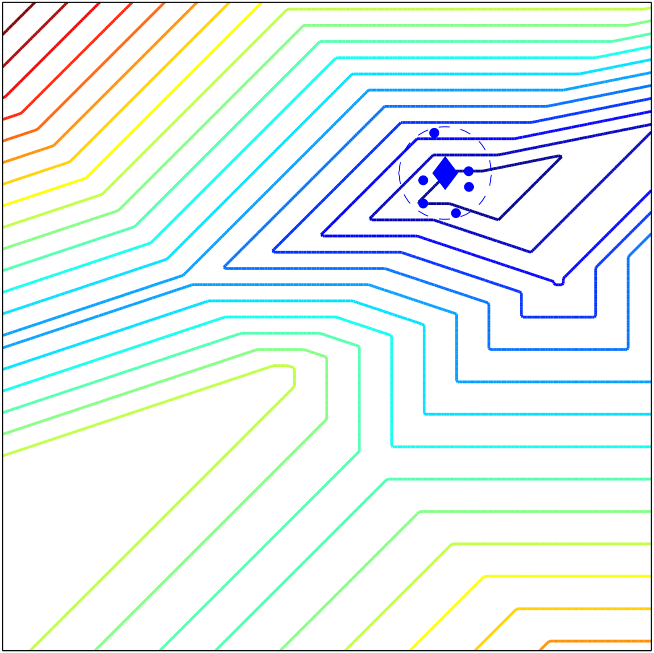


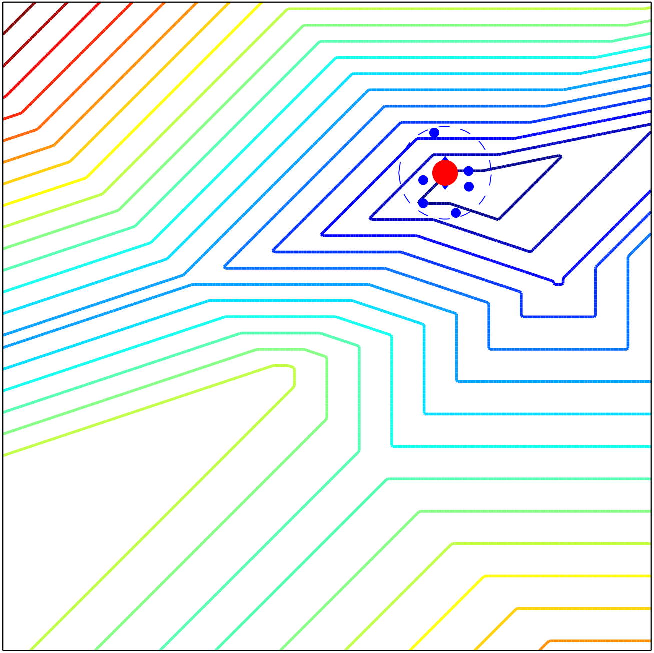












Convergence

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- The same holds for cluster points of the sequence of MS4PL iterates.

Test problems

Let h be a censored ℓ_1-loss function. Given data $d \in \mathbb{R}^p$, censors $c \in \mathbb{R}^p$, and the mapping $F : \mathbb{R}^n \to \mathbb{R}^p$, we define

$$f(x) = \sum_{i=1}^{p} |d_i - \max\{F_i(x), c_i\}| .$$

That is, $\psi = 0$, and

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Define F to be the 53 vector mapping in the Móre and Wild benchmarking set. $2 \leq n \leq 12$, $2 \leq p \leq 45$.

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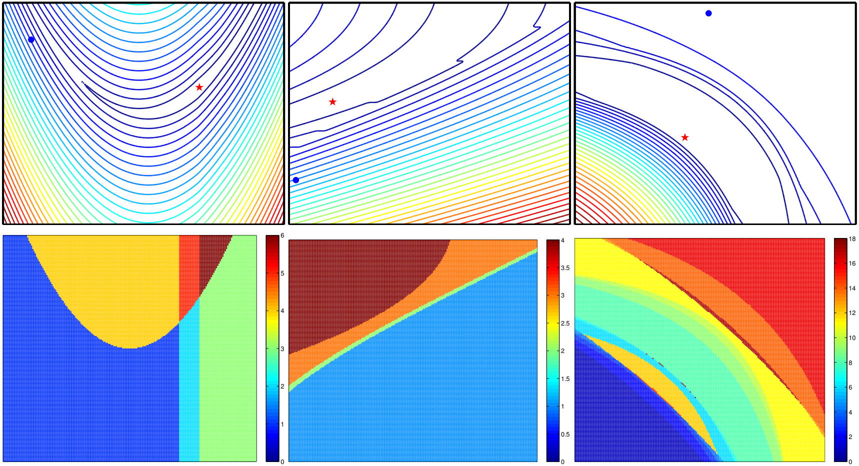
Set $c_1 = -\infty$ and $d_1 = 0$.

Examples



Algorithms to compare

MS4PL-1 Using manifolds at x^k

MS4PL-2 Using manifolds in $\mathcal{B}(x^k, \Delta_k)$

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PLC POUNDERs using a single manifold active at x^k to form a master model

SLQP-GS Gradient sampling algorithm from Curtis

GRANSO BFGS-SQP algorithm Mitchell, Curtis, and Overton. (Can handle constraints too.)

Gradient sampling

Theorem (Rademacher)

If $S \subset \mathbb{R}^n$ is open and $f : S \to \mathbb{R}$ is locally Lipschitz on S, then f is differentiable almost everywhere on S.

Gradient sampling

1. Approximate $\partial f(x^k)$ by sampling $m \geq n+1$ points $x^{k,j}$ in $\mathcal{B}(x^k, \epsilon_k)$. Set

$$\mathfrak{G}^k = \operatorname{conv}\left\{\nabla f(x^{k,1}), \ldots, \nabla f(x^{k,m})\right\}$$

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- Iterates must not be points of nondifferentiability
- Significant sampling may be required

Tests

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$$f(x^0) - f(x^j) \geq (1-\tau)(f(x^0) - \tilde{f}_p)$$

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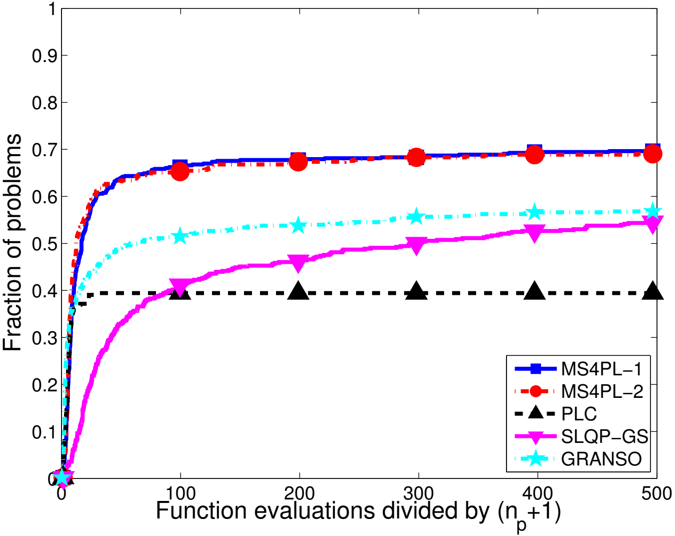
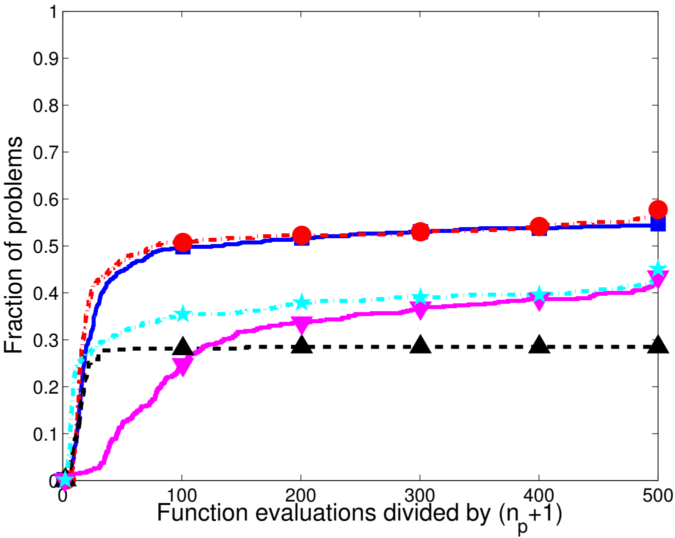
Data profiles

f-test, $\tau = 10^{-2}$



Data profiles

f-**test**, $\tau = 10^{-5}$



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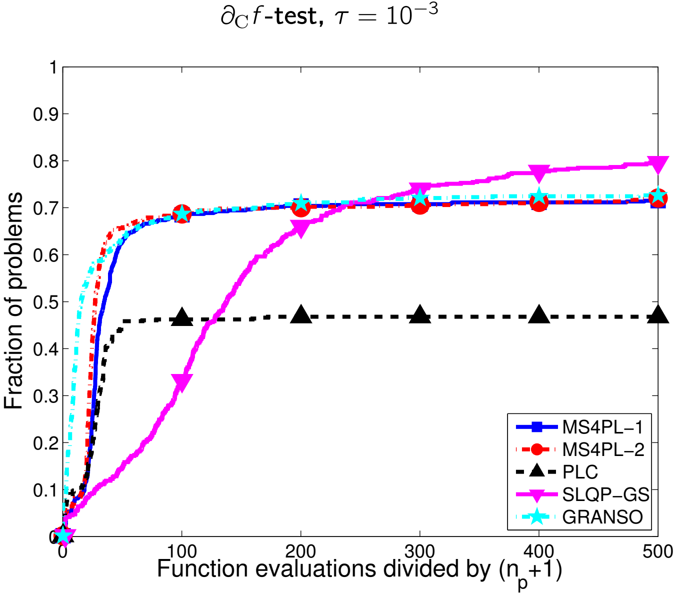

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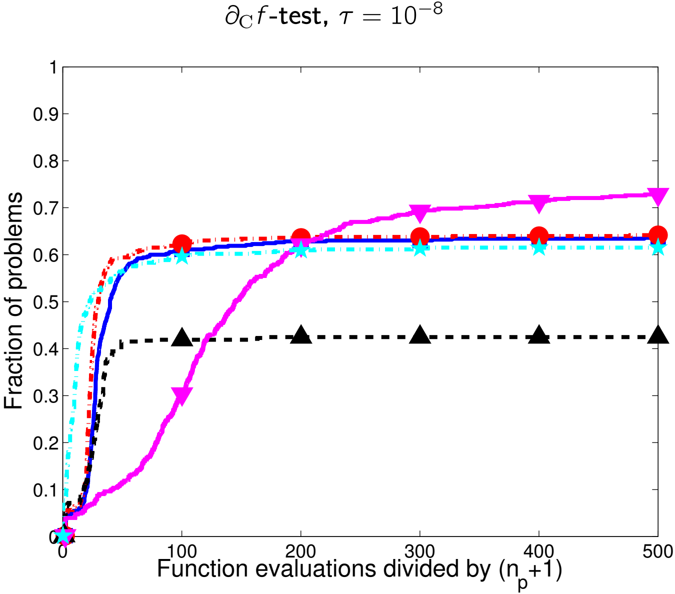

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Email `jmlarson@anl.gov` for a preprint.

Thank you!